Almost Pythagorean

E.E. Ballew, 2010

For some reason I really like little simple geometric relations that remind me of the Pythagorean Theorem. I still remember the first time I saw $3^2 + 4^2=5^2$ set next to $3^3+4^3+5^3=6^3$ and wondered.... Could it be????... Is it possible???? Oh go on, you know you are going to check... how could you resist?

Every high school student is introduced to the Pythagorean Theorem somewhere in their journey from the seventh to the twelfth grade. Few of them, however, realize the wide application and importance of the theorem in mathematics and science. In the words of Arjen Dijksman, one of my favorite bloggers, "Physics makes extensive use of the Pythagorean law relating the squares of the sides of a right triangle. The well-known $a^2 + b^2 = c^2$ relation is of special interest for the determination of distances and lengths of vectors, but also for energy conservation laws and Lorentz transformations." The simple rule is important enough to mathematicians that they sometimes take it to their graves, literally, as did the 19th Century British mathematician Henry Perigal (1801 - 1898) who invented a dissection proof of the Pythagorean Theorem that he was so proud of he used it on his tombstone.

The rule is not just a western tradition. A rule equivalent to the Pythagorean Theorem appeared in the Chinese classic Zhoubi suanjing, "Zhou Shadow Gauge Manual", and was called the Gougu Rule . A "shadow gauge" is the Chinese equivalent of the Greek gnomon, or what we would call a sundial. The work was written sometime between 100 BC and 100 AD. In it the author claims that the Emperor, "… quells floods, deepens rivers and streams, surveys high places and low places by using the Gougu rule." This is the earliest written evidence of the Pythagorean Theorem in China, but legends suggest it may have been known in China prior to the life of Pythagoras.

Students also fail to recognize the many "variations on a theme" that are either related in some mathematical way to the theorem, or just similar enough to be "pretty." Over the years I have come across a few dozen of them, I imagine, and like the irresponsible grasshopper in Aesop's fable, I failed to lay aside notes for the winter, and now I am trying to recall them from the memory of the fleeting summers of my life. These are the ones I can recall now, and if any of you more industrious ant-types out there have others to share, please advise.

I hope, although I fear it is untrue, that at the very least every high school student is introduced to the idea the Pythagorean Theorem so familiar to them in the plane can be generalized to any number of dimensions. For two points in three-space (or four-space or five-space….) , the square of the distance between the two points is equal to the square of the x-differences + the square of the y-differences, plus the square of the z-differences (carry on in like fashion to the higher dimensions of your choice). Thus in a common rectangular box, or room, the distance between opposite corners, when squared, is equal to the sum of the squares of the length, width, and height. The simple beauty of a mathematical equation lets us change the $x^2 + y^2 = d^2$ for a two space distance, into $x^2 + y^2 + z^2 = d^2$ and so on ad infinitum. My more realistic assessment from 30+ years of high school teaching is that many students graduate without ever realizing that the "distance formula" they learned in Alg I and the Pythagorean Theorem they learned (over and over it seems) are the same big idea. In the same way, several of the generalizations and extensions that follow are in some ways, similar, and in at least one case, identical.

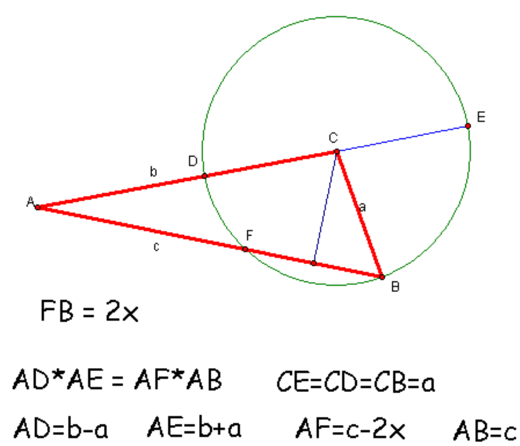
The most well-known generalization of the Pythagorean Theorem is the law of cosines. The Pythagorean Theorem says that $c^2 = a^2 + b^2$ if, and only if, the angle opposite side c is a right angle…that is, it is true if the triangle ABC is a right triangle. The law of cosines simply extends this to triangles with any measure of the angle in question. One of the interesting things about the law of cosines is that it was known a thousand years before the term cosine was created. In book two of Euclid's Elements he describes the property first for the obtuse triangle in Proposition 12, and then for an acute triangle in Proposition 13. Here is the way they are translated in the second edition of A History of Mathematics by Carl Boyer, (revised by Uta Merzbach)

> Proposition 12: In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular toward the obtuse angle.
> Proposition 13. In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides

> about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular toward the acute angle.

The phrase about "twice a rectangle" can be understood to mean two times AC (the side on which the perpendicular falls) times AD (the straight line cut off outside by the perpendicular.) This is the same as our common expression of the law since AD is equal to AB * Cos (BAC). To be more explicit, the area of a rectangle with sides of AC and AD will have an area equal to AC*AB*Cos(BAC).

A common proof of the property in textbooks today is to draw the angle C at the origin and place B at the point (a,0) along the x-axis. This leads to the easy declaration that the coordinates of point A must be at (b*cosC, b*SinC). Then it is easy to show the proof by applying the distance formula for AB (side c) and squaring both sides of the expression and some simple trig identities do the rest.



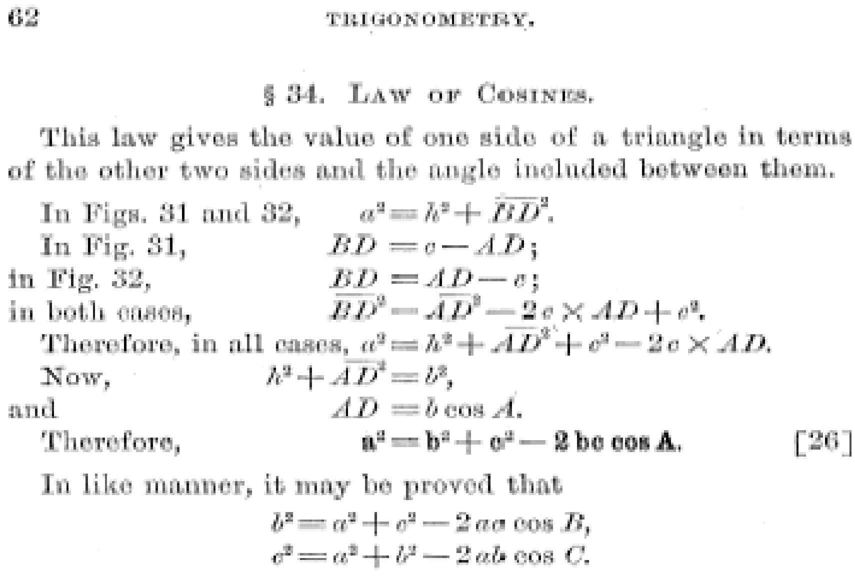
A somewhat prettier proof using only geometry is the proof used by Pitiscus in Trigonometriae sive de triangulorum libri quinque which is illustrated below. (It was Pitiscus, by the way, who first used the word trigonometry in 1595.. Edmund Gunter created the terms "co.sinus" and "co.tangens", one of which was quickly modified to "cosinus" by John Newton around 1660. By 1675 Sir Jonas Moore had abbreviated this down to "cos".)

From AD*AE = AF*AB we replace terms to get
(b-a)(b+a)= c(c-2x)
$b^2 - a^2 = c^2 - 2cx$
and reordering terms gives $b^2 = a^2 + c^2 - 2cx$. A quick look at the diagram shows that x is equal to a Cos B, and with this substitution (not made by Pitiscus) we get the modern form, $b^2 = a^2 + c^2 - 2ca\cos(B)$.

According to Jeff Miller's web site on the First use of some mathematical terms, the application of the name "Law of Cosines" was near the end of the 19th century;

LAW OF COSINES is found in 1895 in Plane and spherical trigonometry, surveying and tables by George Albert Wentworth: "Law of Cosines. ... The square of any side of a triangle is equal to the sum of the squares of the other two sides, diminished by twice their product into the cosine of the included angle."

Here is an image of the page.


62 TRIGONOMETRY.

§ 34. LAW OF COSINES.

This law gives the value of one side of a triangle in terms of the other two sides and the angle included between them.

In Figs. 31 and 32, $a^2 = h^2 + \overline{BD}^2$.

In Fig. 31, $BD = c - AD$;

in Fig. 32, $BD = AD - c$;

in both cases, $\overline{BD}^2 = \overline{AD}^2 - 2c \times AD + c^2$.

Therefore, in all cases, $a^2 = h^2 + \overline{AD}^2 + c^2 - 2c \times AD$.

Now, $h^2 + \overline{AD}^2 = b^2$,

and $AD = b \cos A$.

Therefore, $\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}$. [26]

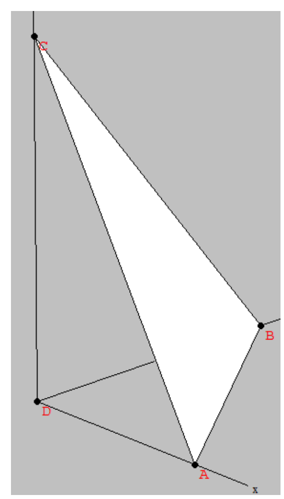
In like manner, it may be proved that

$b^2 = a^2 + c^2 - 2ac \cos B$,

$c^2 = a^2 + b^2 - 2ab \cos C$.

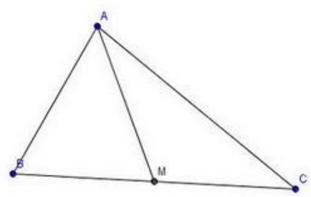
I am still searching for an earlier use of the phrase. It seems unusual, to me at least, that its first appearance was as a title. The formula, exactly as we might write it today, appears in the trigonometry addendum (pg 305) at the end of John Playfair's 1804 edition of Elements of Geometry which I have on my bookshelf.

There is also a spherical law of cosines, and just as the Pythagorean theorem is a simplification of the law of cosines, there is a simplification of the spherical law of cosines that produces a relationship between the sides of a spherical right triangle. In spherical triangles both sides and angles are usually treated by their angle measure since sides are arc lengths of a great circle. Using capital letters to represent angles, and lower case to represent the opposite sides, the law for sides is given as cos c = cos a cos b + sin a sin b cos C . When angle C is a right angle, then Cos(C) = 0, and the "Pythagorean relation for spherical right triangles" is given by cos c = cos a cos b.



Another 3-d analogy of the Pythagorean Theorem is de Gua's Theorem. If a tetrahedron has all right triangles for the three faces meeting at one vertex (D in the image at right), de Gua showed that if the areas of the three right angled triangles are A, B, and C, and the triangle on the base has area D, then $A^2+B^2+C^2=D^2$. Notice that these squares are the squares of areas, so this is a fourth degree variation. The theorem is named for J. P. de Gua de Malvex but it seems it had been previously known to both Descartes and Faulhaber.

My friend Dave Renfro recently sent me an article that reminded me of a few of the old relations I used to point out at NCTM lectures.

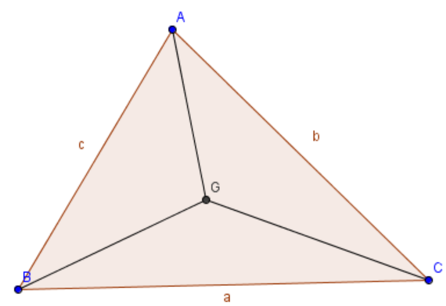


The Two Triangles formed when the median is drawn to any side are almost Pythagorean Treat the two sides not cut by the median as if they were the hypotenii (hypotenuses?) of a right triangle. Both are wrong, but in sum, they are right..

That is, while neither of the following are true, $AM^2+MC^2= AC^2$ or $AM^2+MB^2=AB^2$; Adding the two equations produces a true relation.
$AM^2+MC^2+AM^2+MB^2= AC^2+AB^2$

The proof is pretty easy using the Law of Cosines.

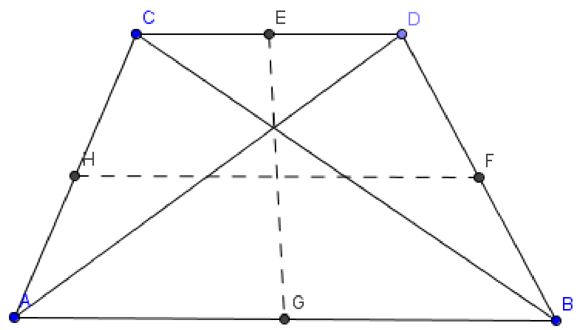
Another and about as easy to prove....
Let G be the centroid of Triangle ABC, then $AB^2+BC^2+CA^2 = 3 (GA^2+GB^2+GC^2)$



For teachers of geometry, it might be a good challenge for your students to show that a necessary and sufficient condition for a parallelogram is that the sum of the squares of the diagonals is equal to the sum of the squares of the sides. Perhaps more interesting, if the quadrilateral is a trapezoid, then the “almost Pythagorean” relation becomes the sum of the squares of the diagonals is equal to the sum of the squares of the non-parallel sides, plus twice the product of the parallel bases (perhaps more of a law-of-cosines look-a-like). This last appears as exercise 20 on page 143 of “The Elements of Geometry, after Legendre”, by Charles Scott Veneble in 1881.

While looking up some historical notes only a day ago, I came across Charles Hutton’s “Philosophical and Mathematical Dictionary”, written in 1815. Hutton is the one who, in an earlier addition of this same book, confused or confounded the use of the terms trapezoid and trapezium. For those who did not know, the words trapezoid and trapezium have different meanings in the US and much of the rest of the English speaking world. Both come originally from the Greek word for table. Today, in the USA, the term trapezoid refers to a quadrilateral with one pair of sides parallel and a trapezium to one with NO parallel sides. Actually, the term for the case with no parallel sides is almost never used, so trapezium is an archaic term at best in the US. This is exactly the reverse of the original meanings and the meanings in some countries, particularly England, today. Here is a short comment on how this came about from Jeff Miller, a teacher at Gulf High School in New Port Richey, Florida, who maintains an excellent page on the first use of some common mathematical terms:

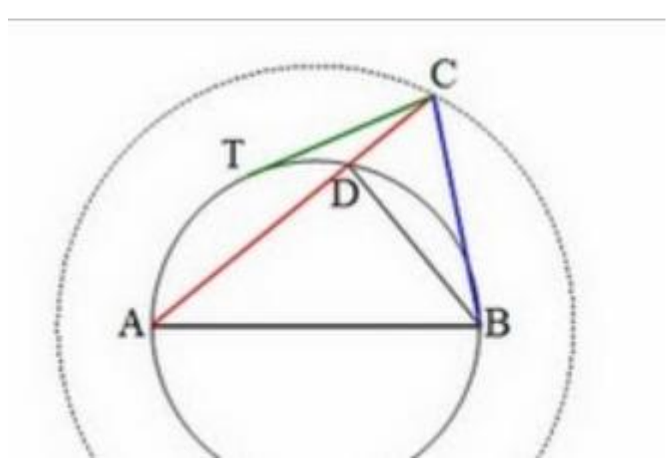
> "TRAPEZIUM and TRAPEZOID. The early editions of Euclid 1482-1516 have the Arabic helmariphe; trapezium is in the Basle edition of 1546. Both trapezium and trapezoid were used by Proclus (c. 410-485). From the time of Proclus until the end of the 18th century, a trapezium was a quadrilateral with two sides parallel and a trapezoid was a quadrilateral with no sides parallel. However, in 1795 a Mathematical and Philosophical Dictionary by Charles Hutton (1737-1823) appeared with the definitions of the two terms reversed: Trapezium...a plane figure contained under four right lines, of which both the opposite pairs are not parallel. When this figure has two of its sides parallel to each other, it is sometimes called a trapezoid. No previous use of the words with Hutton's definitions is known. Nevertheless, the newer meanings of the two words now prevail in U. S. but not necessarily in Great Britain (OED2).



In Hutton's book he points out that if you take a *trapezium* ABCD and label the midpoints of each side as EFGH in order, then the sum of the squares of the two diagonals is equal to the twice the sum of the squares of the two medials, or, $AD^2 + BC^2 = 2(EG^2 + FH^2)$.

Of course if the quadrilateral is cyclic (all four vertices lie on a single circle, or opposite angles are supplementary) then we can say that the product of the diagonals is equal to the sum of the products of the opposite sides.

Very recently I found another example on the blog of Arjen Dijksman, mentioned at the beginning of this article. If we draw a circle with one leg of triangle ABC as the diameter, then $AB^2 = AC^2+BC^2- 2\ CT^2$.



This is, of course, too close to the law of cosines to be ignored.. and it is easy to see that $t^2 = a*b*\cos(C)$... in fact as soon as we set t^2=AC*DC, and recognize that angle BDC is a right angle, we have Cos(C)= DC/BC... thus AC*DC = AC*BC*(DC/BC)= ab Cos(C).

Recently I received a note from Professor Robin Whitty, who maintains a really nice math site called “Theorem of the Day” at http://www.theoremoftheday.org/ that reminded me of another Pythagorean look alike, Des Carte’s Circle Theorem.

The theorem, sometimes called the "Kissing Circles Theorem", is about the relationship of the radii of four mutually tangent circles. Modern mathematicians have adopted the a notation of "bend" to replace the reciprocal of the radii in Des Carte’s theorem (it seems nobody likes fractions anymore. For each circle let the "bend" equal the reciprocal of the radius, then $1/r_1 = b_1$. With this notation the formula can be written as $\left(b_1^2 + b_2^2 + b_3^2 + b_4^2\right) = \frac{1}{2}\left(b_1 + b_2 + b_3 + b_4\right)^2$. Or in words, the sum of the squares of the bends is equal to one-half the square of the sum of the bends.

The theorem is also sometimes called Soddy’s theorem because in 1936 Sir Fredrick Soddy rediscovered the theorem again, and then proceeded to write a really interesting poem about the relationship called “The Kiss Precise”.

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,

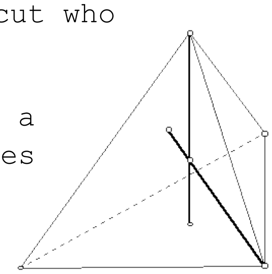
The sum of the squares of all four bends
Is half the square of their sum.

Soddy may also be known to students of science for receiving the Nobel Prize for Chemistry in 1921 for the discovery of the decay sequences of radioactive isotopes. According to Oliver Sacks' wonderful book, Uncle Tungsten, Soddy also created the term "**isotope**" and was the first to use the term "chain reaction". In a strange "chain reaction" of ideas, Soddy played a part in the US developing an atomic bomb. Soddy's book, The Interpretation of Radium, inspired H G Wells to write The World Set Free in 1914, and he dedicated the novel to Soddy's book. Twenty years later, Wells' book set Leo Szilard to thinking about the possibility of Chain reactions, and how they might be used to create a bomb, leading to his getting a British patent on the idea in 1936. A few years later Szilard encouraged his friend, Albert Einstein, to write a letter to President Roosevelt about the potential for an atomic bomb. The prize-winning science-fiction writer, Frederik Pohl, talks about Szilard's epiphany in Chasing Science (pg 25),

".. we know the exact spot where Leo Szilard got the idea that led to the atomic bomb. There isn't even a plaque to mark it, but it happened in 1938, while he was waiting for a traffic light to change on London's Southampton Row. Szilard had been remembering H. G. Well's old science-fiction novel about atomic power, *The World Set Free* and had been reading about the nuclear-fission experiment of Otto Hahn and Lise Meitner, and the lightbulb went on over his head."

Thumbing through some old journal articles sent to me by David Renfro, I discovered yet another "almost Pythagorean" relation. In the "Problems and Solutions" section of the American Mathematical Monthly from May of 1934 it gives a proof of a problem previously submitted by J. Rosenbaum from the Milford School in Milford Connecticut who had asked the readers to :

> "Prove that the sum of the squares of the medias of a tetrahedron equals four-ninths the sum of the squares of the edges."

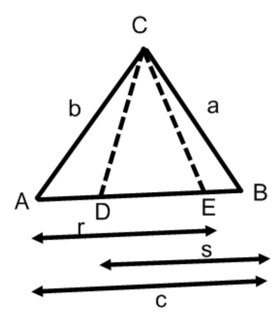


The median of a tetrahedron (*I have previously suggested the term "medial" or "medial segment"*) is a segment from a vertex to the centroid of the opposite side. The readers provided the proof; it's true, and "almost Pythagorean"

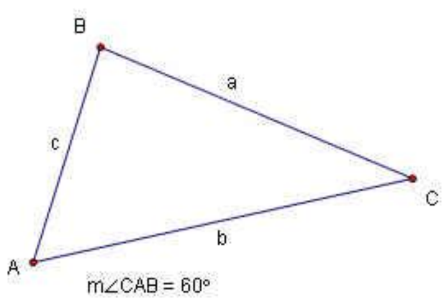
This next one is not just Pythagorean-like, it is actually a variant of the Pythagorean Theorem. I was reminded of this one on a visit to

Alexander Bogomolny's Cut-the-knot Wiki site. For angles A, B, and C in a triangle, $\text{Cos}^2(A)+\text{Cos}^2(B)+\text{Cos}^2(C)=1$ if, and only if, ABC is a right triangle. It is easy to prove it is true if any one of the angles is a right angle. Suppose C is a right angle; then Cos(C)=0 so we have $\text{Cos}^2(A)+\text{Cos}^2(B)+0=1$ or just $\text{Cos}^2(A)+\text{Cos}^2(B)=1$. Now keep in mind that if C is a right angle, then A+B= 90° (or pi radians as you choose). So Sin A = Cos B and vice-versa. Using $\text{Sin}^2(A)$ in place of $\text{Cos}^2(B)$ we get the well known identity that $\text{Sin}^2(A)+\text{Cos}^2(A) = 1$. To prove the *only if*, we begin with a general truth about triangles, that $\text{Cos}^2(A)+\text{Cos}^2(B)+\text{Cos}^2(C)+2\text{Cos}(A)\text{Cos}(B)\text{Cos}(C)=1$. For the sum of the squares to be equal to one, it must be true that 2Cos(A)Cos(B)Cos(C)=0, which can only happen if one of Cos(A), Cos(B) or Cos(C)=0… and thus one of them must be a right angle.

Here is one I picked up from Wikipedia that I had never known. I'm not sure how commonly it is known. Did you know? In any triangle, ABC, if you make an isosceles triangle ADE where D and E are on the segment BC, and the two base angles at D and E are both congruent to angle A of ABC. For ease of notation, letting the distance from A to E be r, and from B to D be s, then for triangle sides a, b, and c it is true that $a^2 + b^2 = c(r+s)$. If the angle at A is approaches a right angle, then the base of the isosceles triangle gets smaller and smaller and s+r approaches c as a limit.



The Pythagorean Theorem is a special relationship that exists between the sides of triangles when one angle is 90°. This next example is a special rule when the triangle has one an angle of 60°.



If we let the angle at vertex A be the sixty degree angle, then we can show that

$$\frac{a^3+b^3+c^3}{a+b+c}=a^2$$

If we begin with the law of cosines, we know that $a^2 = b^2 + c^2 - 2bc$ $\text{Cos}(A)$… but if A is a sixty degree angle, then Cos (A) = ½, and the last can be simplified down to $a^2 = b^2 + c^2 - bc$ Now if we multiply both sides by (b+c), we arrive at $a^2(b+c) = (b+c)(b^2-bc+c^2)$ which makes the right hand side equal $b^3+ c^3$ …….. to give us $a^2(b+c) = b^3 + c^3$.

Now if we add a^3 to both sides, we are nearly home… $a^3 + a^2(b+c) = a^3 + b^3 + c^3$. Artfully rearranging the left side into $a^2(a)+ a^2(b+c)$ we can factor the a^2 to make the left side $a^2 (a+b+c) = a^3+b^3+c^3$…. And with one last division, we have established the relationship we seek…
$a^2 = \frac{a^3+b^3+c^3}{a+b+c}$.

The proof that this is an “if and only if” relationship simply proceeds in the reverse order.