A Detailed and Elementary Solution to $x^{17} = 1$

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1. Introduction

This manuscript was written during a three week period in March-April 2011 and it was motivated by Pat Ballew's 20 March 2011 blog entry *Gauss and Constuctable Polygons* <**http://pballew.blogspot.com/**>. It occurred to me that, in all of the many dozens of expositions about solutions to $x^{17} = 1$ I had encountered in 30+ years, none gave all the details for obtaining one of the lengthy square root expressions one often sees exhibited. This includes the huge number of class notes and unpublished manuscripts that have populated the internet in the past 15 years, although that may no long be the case after this manuscript circulates. I suppose a case could be made that this is done in Klein [45] (Article 6, pp. 29-32) and in Barnard/Child [4] (pp. 174-175). However, I have not encountered any treatment where sufficient detail was given so that someone competent in high school algebra and trigonometry, but otherwise not very mathematically sophisticated, could be reasonably expected to follow. With this in mind, there are three main ways that I have tried to distinguish the present treatment from the many already in existence.

First, I have tried to be extremely detailed and explicit in the exposition, especially in carrying out algebraic manipulations, in rewriting numerical expressions, and in giving precise explanations for how the manipulations and rewriting are done.

Second, in the case of obtaining explicit square root expressions related to the equation $x^{17} = 1$, no computer algebra systems or calculators are used. In fact, I believe the only instance in which I needed to multiply something out by hand using grade school arithmetic methods was (34)(170) = 578 at one point, although even this could have been avoided, as will be evident when I obtain a more compact square root expression for $\cos\left(\frac{2\pi}{17}\right)$.

Third, I have paid as much attention to the completeness and accuracy of the bibliography as I have with the mathematics. Thus, author names, journal titles, page citations, and the like are detailed and explicit. I have also given careful thought to the items included, with the exception of the $x^{257} = 1$ entries (whose novelty and rarity merit the exception). My intended audience consists mainly of English readers who are high school students and their teachers, faculty at small colleges, and other math enthusiasts who do not have access to the resources of a research university library (an extensive book and journal collection, JSTOR and other expensive digital collections, etc.). Thus, the items in the bibliography were mostly selected for being freely available on the internet and for assuming relatively little mathematical background knowledge. I made a few exceptions to these principles, but this was done when I thought an item might be especially useful to someone interested in exploring the subject further. Also, I have noted which books have been reprinted by Dover Publications, because such a reprinting (even if no longer in print) makes it much more likely that the book can be found on the shelves of a public library or a small college library.

2. Constructible Numbers

The arithmetic/algebraic idea of a constructible real number can be defined in the following way.

A real number is *constructible* if it can be obtained in a finite number of steps, beginning with the integers (in fact, we can always begin with just the number 1), such that each step consists of one of the following 5 operations: adding, subtracting, multiplying, or dividing two (not necessarily distinct) numbers obtained previously (not necessarily from the same previous step), or by taking the positive square root of a number obtained in a previous step.

Remark: Starting with just 1, we can reach any given positive integer in a finite number of steps (the minimum number of steps can vary with the positive integer, of course): $1 + 1 = 2$, $1 + 2 = 3$, $2 + 2 = 4$, etc. Moreover, we can obtain 0 (e.g. $1 - 1 = 0$), and then using 0 we can obtain each negative integer (e.g. $0 - 4 = -4$). Thus, if we begin with just the number 1, then by including steps at the beginning that lead to any later needed integer, we can obtain a finite sequence of admissible steps that leads to any specified constructible number by beginning with just the number 1. Although this simplification allows certain results to be more efficiently proved, it will not be important to us. However, I thought this discussion would be useful in illustrating how certain aspects of the definition can be used.

Example 1: The constructability of $\sqrt{5} - \sqrt{2}$ can be verified by the sequence

$$2,\ 5 \rightarrow \sqrt{2} \rightarrow \sqrt{5} \rightarrow \sqrt{5} - \sqrt{2}$$

Example 2: The constructability of $\sqrt{2 + 5\sqrt{3}} + \sqrt{2 - 5\sqrt{3}}$ can be verified by the sequence

$$2,\ 3,\ 5 \rightarrow \sqrt{3} \rightarrow 5\sqrt{3} \rightarrow 2 + 5\sqrt{3} \rightarrow 2 - 5\sqrt{3} \rightarrow \sqrt{2 + 5\sqrt{3}} \rightarrow \sqrt{2 - 5\sqrt{3}}$$
$$\rightarrow \sqrt{2 + 5\sqrt{3}} + \sqrt{2 - 5\sqrt{3}}$$

Basically, any real number that can be written as an explicit square root expression is constructible. Conversely, any constructible real number can be written as an explicit square root expression. Sometimes it is not obvious that a real number is constructible. For example, $\sqrt[3]{\sqrt{243} + \sqrt{242}} - \sqrt[3]{\sqrt{243} - \sqrt{242}}$ is constructible, since this number is equal to $2\sqrt{2}$ (an equality I challenge the reader to prove).

An alternate definition of *constructible real number* can be given by replacing "taking the positive square root of a number obtained in a previous step" with "solving a quadratic equation with coefficients obtained in previous steps". It is not difficult to see that these two definitions give rise to the same collection of real numbers.

If a number is constructible by the first definition, then it is constructible by the second definition. To see this, note that the step $c \to \sqrt{c}$ (for $c > 0$) can be replaced with the steps $c \to x^2 - c = 0 \to \sqrt{c}$.

If a number is constructible by the second definition, then it is constructible by the first definition. To see this, note that the steps

$ax^2 + bx + c = 0 \to \frac{-b + \sqrt{b^2 - 4ac}}{2}, \frac{-b - \sqrt{b^2 - 4ac}}{2}$ can be replaced with the steps (several of which I've consolidated into single steps)

$$a, b, c \to -\frac{b}{2a}, \pm\frac{1}{2a} \to \sqrt{b^2 - 4ac} \to \pm\frac{1}{2a}\sqrt{b^2 - 4ac} \to -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$$

Therefore, we can describe the constructible real numbers as those real numbers that, starting with the integers, can be obtained by solving a finite number of quadratic equations (in series and/or in parallel, to use electric circuit terminology). In this sense, a number being constructible is somewhat like a relaxed notion of a number being rational, since the rational numbers are those numbers that can be obtained by solving linear equations, beginning with the integers. In the case of rational numbers, however, you can always manage with at most one linear equation, but this is not the case with the constructible numbers.

Here are two examples of how the second definition can be used to define a constructible number. A more elaborate example is given in Section 6.

Example 3: Let a be the positive solution of $4x^2 + x - 1 = 0$. Let b be the positive solution of $x^2 + 4ax - 1 = 0$. Let c be the negative solution of $x^2 + (4a + 1)x - 1 = 0$. Let d be the negative solution of $x^2 + bx + c = 0$, divided by -2.

Example 4: Let p, n be the positive and negative solutions of $x^2 + x - 4 = 0$. Let r be the positive solution of $x^2 - px - 1 = 0$. Let s be the positive solution of $x^2 - nx - 1 = 0$. Let t be the greatest solution of $x^2 - rx + s = 0$.

One can show that $d = \cos\left(\frac{\pi}{17}\right)$ by an analysis of the results obtained in Brown [35].

One can show that $t = \cos\left(\frac{2\pi}{17}\right)$ by using the results in our Section 3.

There are two natural ways to define what it means for a complex number to be constructible. One way is to use either of the definitions above (closure under the $\sqrt{\ }$ operation or closure under solving quadratic equations), and keep all *complex numbers* that are generated, instead of just keeping all real numbers that are generated. In the same way as above, it is not difficult to see that closure under the $\sqrt{\ }$ operation and closure under solving quadratic equations each gives rise to the same collection of complex numbers. (Note that the quadratic formula still holds when the coefficients of a quadratic equation are complex numbers.) The other way is to use our definition of what it means for a real number to be constructible, and then define $a + b\mathrm{i}$ (where a, b are real numbers) to be constructible if and only if both a and b are constructible real numbers. We will show that these two definitions lead to the same collection of complex numbers.

We first show that if $a + b\mathrm{i}$ is constructible by the second definition, then $a + b\mathrm{i}$ is constructible by the first definition. Since $a + b\mathrm{i}$ is constructible by the second definition, we know that a and b are constructible real numbers. Hence, each of a and b are *complex numbers* that are constructible by the *first definition*. Since $\mathrm{i} = \sqrt{-1}$ is constructible by the first definition, it follows that the product $b\mathrm{i}$ is constructible by the first definition. Finally, now that we have both a and $b\mathrm{i}$ constructible by the first definition, it follows that the sum $a + b\mathrm{i}$ is constructible by the first definition.

We next show that if a complex number is constructible by the first definition, then it is constructible by the second definition, which will complete our proof. This next result will follow easily from the observation that when each of the operations of addition, subtraction, multiplication, division, and square root is performed on complex numbers, the real and imaginary parts of the output can be obtained by using only these operations on the real and imaginary parts of the input(s). For example, $(a_1 + b_1\mathrm{i}) + (a_2 + b_2\mathrm{i}) = (a_1 + a_2) + (b_1 + b_2)\mathrm{i}$ and $\dfrac{a_1 + b_1\mathrm{i}}{a_2 + b_2\mathrm{i}} = \left[\dfrac{a_1a_2 + b_1b_2}{(a_2)^2 + (b_2)^2}\right] + \left[\dfrac{a_2b_1 - a_1b_2}{(a_2)^2 + (b_2)^2}\right]\mathrm{i}$. Moreover, if $z^2 = a + b\mathrm{i}$, then

$$z = \begin{cases} \pm(\alpha + \beta\mathrm{i}) & \text{if } b > 0 \\ \pm(\alpha - \beta\mathrm{i}) & \text{if } b < 0, \end{cases}$$

where $\alpha = \sqrt{\frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right)}$ and $\beta = \sqrt{\frac{1}{2}\left(-a + \sqrt{a^2 + b^2}\right)}$.

Now suppose that a certain complex number is constructible by the first definition. Then the number can be obtained by a sequence of allowable steps, beginning with the integers. Each integer has the property of having real and imaginary parts that are constructible real numbers, and by what we just demonstrated (explicitly, in the case of addition, division, and square root), each of the allowable steps preserves this property. Therefore, after all the allowable steps are performed, we will get a complex number whose real and imaginary parts are constructible real numbers, and hence this complex number is constructible by the second definition. (A more rigorous approach would format this proof by making use of mathematical induction.)

Perhaps the simplest example of a real number that is not constructible is $\sqrt[3]{2}$. For relatively elementary proofs that $\sqrt[3]{2}$ is not constructible, see Courant/Robbins [12] (pp. 134-135) and Hadlock [21] (pp. 24-26). The non-constructability of $\sqrt[3]{2}$ is historically important because it implies the Greek problem of duplicating the cube is not geometrically solvable by ruler and compass methods.

A more general result about cubic equations is not much harder to prove, and one often finds specific proofs that $\sqrt[3]{2}$ is not constructible omitted in favor of proving the following result about cubic equations. If a cubic equation with rational coefficients has no rational solutions (checking this condition gives a nontrivial application of the rational roots theorem in high school algebra and precalculus courses), then the cubic equation does not have a constructible solution. Since $x^3 - 2 = 0$ has no rational solutions, it follows that $\sqrt[3]{2}$ is not constructible. Also, from the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, it follows that if $x = \cos 20^\circ$, then $8x^3 - 6x - 1 = 0$. Since this is a cubic equation having no rational solutions (use the rational root test), it follows that $\cos 20^\circ$ is not a constructible number. The non-constructability of $\cos 20^\circ$ is historically important because it implies that the Greek problem of trisecting an angle is not geometrically solvable by ruler and compass methods. For proofs of the cubic equation result, see Courant/Robbins [12] (pp. 136-137), Dickson [16] (pp. 221-222), Dickson [17] (pp. 32-34), Dickson [18] (pp. 33-36), Kazarinoff [23] (pp. 53-54), Meschkowski [28] (pp. 119-121), and Meyerson [29] (pp. 572-573).

Finally, there is an even more general result that implies the previous result about cubic equations: If a polynomial with rational coefficients is irreducible over the rational numbers (this means that the polynomial cannot be factored into two polynomials, each of lower degree and each with rational coefficients) and the degree of the polynomial is not a power of 2 (that is, the degree is not equal to 1, 2, 4, 8, 16, etc.), then the polynomial has no constructible zeros. For proofs of this result, see Davis [13], Dickson [14] (pp. 357-363), Lovitt [26] (pp. 205-210), Papantonopoulou [30] (p. 349), and Klein [45] (Chapter 1, pp. 5-12).

Some Remarks on Conic Constructible Numbers

If we alter the definition of constructible number so that, besides solutions to quadratic equations, we also allow solutions to cubic equations (equivalently, we also allow the cube root operation; the nontrivial half of this equivalence follows immediately from the cubic equation formula), then we get a more inclusive notion of constructability that has been studied a lot in recent years. I am using the phrase *conic constructible* (analogous to the phrase *ruler and compass constructible*) because, in the case of positive real numbers, these are exactly the numbers that will be geometrically constructible if intersections involving lines and conic curves are allowed, rather than just intersections involving lines and circles. Incidentally, the Greeks discovered several ways of geometrically solving the problems of duplicating the cube and trisecting an angle by allowing the use of intersections involving lines and conic curves. A list of 20 references involving conic constructible numbers is given at <**http://tinyurl.com/4hhbnb9**>.

For which values of n are all solutions to $x^n = 1$ ruler and compass constructible?

The values of n are exactly those numbers having the form $2^r p_1 p_2 \ldots p_k$, where $r \geq 0$, $k \geq 0$, and the integers $p_1, p_2, \ldots, p_k$ are *distinct* prime numbers greater than 2 such that each of the primes can be written as $2^i + 1$ for some integer $i > 0$. At this time the only such prime numbers known are 3, 5, 17, 257, and 65537 (these are called *Fermat primes*). Below is a list of the 26 positive integers n less than 100 such that all solutions to $x^n = 1$ are ruler and compass constructible. (This is sequence A003401 at *The On-Line Encyclopedia of Integer Sequences*.)

1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48,
51, 60, 64, 68, 80, 85, 96, ...

For which values of n are all solutions to $x^n = 1$ conic constructible?

The values of n are exactly those numbers having the form $2^r 3^s p_1 p_2 \ldots p_k$, where $r \geq 0$, $s \geq 0$, $k \geq 0$, and the integers $p_1, p_2, \ldots, p_k$ are *distinct* prime numbers greater than 3 such that each of the primes can be written as $2^i 3^j + 1$ for some (not necessarily distinct) integers $i > 0$ and $j \geq 0$ (see Papantonopoulou [30], p. 421). Below are the 35 *additional integers* we need to include in the above list to obtain the positive integers n less than 100 such that all solutions to $x^n = 1$ are conic constructible. (These values are given at the bottom of p. 83 of Pierpont [31].)

7, 9, 13, 14, 18, 19, 21, 26, 27, 28, 35, 36, 37, 38, 39, 42, 45,
52, 54, 56, 57, 63, 65 70, 72, 73, 74, 76, 78, 81, 84, 90, 91, 95, 97

I'll end this section with the comment that no new real numbers arise (and hence, no new values of n) if we allow the solving of 4th degree equations in addition to solving quadratic and cubic equations. On the other hand, allowing the solving of 5th degree equations will introduce new numbers, and it also introduces additional values of n to the list above (but this larger list will still not be all of the positive integers). Incidentally, allowing the solving of 5th degree equations introduces more new numbers than simply allowing the use of the operations $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, and $\sqrt[5]{}$. In fact, allowing for the solving of 5th degree equations introduces some numbers that we would still not be able to obtain in finitely many steps (beginning with the integers), where each step is an application of one of the four arithmetic operations or one of the infinitely many possible root extraction operations $\sqrt{}$, $\sqrt[3]{}$, ..., $\sqrt[n]{}$, ...

3. A Detailed Algebraic Solution to $x^{17} = 1$

What follows is based on the outline given in Chepmell [39].

Dividing both sides of $x^{17} = 1$ by x gives $x^{16} = \frac{1}{x}$, from which we also get $\frac{1}{x^{16}} = x$.

Adding the last two equations gives $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$.

The last equation is equivalent to $x^{32} - x^{17} - x^{15} + 1 = 0$, which has 32 solutions when taking into account possible repeated roots, while $x^{17} = 1$ has only 17 solutions. Therefore, it seems likely that we have introduced some extraneous solutions by adding equations. We will revisit this issue later (see DIGRESSION below). Incidentally, adding *linear equations* will not introduce extraneous solutions (indeed, adding equations is a common method for solving simultaneous linear equations, a method that gives rise to the matrix row-reduction method in linear algebra), but adding *nonlinear equations* can introduce extraneous solutions (for a simple example, add the equations $x = 0$ and $x^2 = 0$).

We now make some variable substitutions, whose forms are possibly motivated by the fact that $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$ is a *reciprocal equation* (an equation with the property that if r is a solution, then $\frac{1}{r}$ is also a solution; the phrase *recurring equation* was often used in early and mid 1800s). Note that each of the following variable substitutions establishes a quadratic relationship between the new variable and the equation's first term. For example, $x^4 + \frac{1}{x^4} = c$ (the 3rd displayed equation below) is equivalent to $\left(x^4\right)^2 - c\left(x^4\right) + 1 = 0$, which is quadratic in x^4.

$$a = x + \frac{1}{x}$$
$$b = x^2 + \frac{1}{x^2}$$
$$c = x^4 + \frac{1}{x^4}$$
$$d = x^8 + \frac{1}{x^8}$$

Next, note that squaring both sides of $a = x + \frac{1}{x}$ gives $a^2 = x^2 + 2 + \frac{1}{x^2}$. Using $b = x^2 + \frac{1}{x^2}$, this last equation becomes $a^2 = 2 + b$. In the same way, squaring both sides of each of the other displayed equations above leads to the following identities involving a, b, c, and d:

(1) $a^2 = 2 + b$
(2) $b^2 = 2 + c$
(3) $c^2 = 2 + d$
(4) $d^2 = 2 + a$

In the case of equation (4), I made use of the fact that $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$ (see the beginning of this section) and $b = x^2 + \frac{1}{x^2}$.

Equation (1) minus equation (3) gives:

(5) $a^2 - c^2 = b - d$

Equation (2) minus equation (4) gives:

(6) $b^2 - d^2 = c - a$

Now multiply equations (5) and (6), factor, and divide both sides by common factors:

$$\left(a^2 - c^2\right)\left(b^2 - d^2\right) = (b - d)(c - a)$$
$$(a + c)(a - c)(b + d)(b - d) = (b - d)(c - a)$$
$$(a + c)(a - c)(b + d)(b - d) = -(b - d)(a - c)$$

(7) $(a + c)(b + d) = -1$

Next, we define e as follows:

(8) $e = a + c$

Using equation (8), equation (7) becomes $e(b + d) = -1$, which can be rewritten as:

(9) $b + d = -\frac{1}{e}$

Incidentally, there is no possibility of dividing by zero, since equation (7) implies that $a + c$, and hence e, is nonzero.

Adding equations (1) and (3) gives:

(10) $a^2 + c^2 = 4 + b + d$

Adding equations (2) and (4) gives:

(11) $b^2 + d^2 = 4 + a + c$

At this point we turn our attention to obtaining expressions for $(a - c)^2$ and $(b - d)^2$ in terms of e. This will allow us to accomplish two things. First, it will allow us to obtain an equation involving only e, which we will solve to find explicit expressions for the values of e. Second, it will allow us to express a in terms of e, which we will use to find explicit expressions for the values of a by making use of the expressions for e that we found. Once we have obtained explicit expressions for the values of a, we can then obtain explicit expressions for the values of x such that $x^{17} = 1$ by solving $a = x + \frac{1}{x}$.

Using equation (9) to rewrite equation (10) gives $a^2 + c^2 = 4 - \frac{1}{e}$. Therefore, $2a^2 + 2c^2 = 8 - \frac{2}{e}$, and hence from the algebraic identity $(a - c)^2 + (a + c)^2 = 2a^2 + 2c^2$, we get $(a - c)^2 = 2a^2 + 2c^2 - (a + c)^2 = 8 - \frac{2}{e} - (a + c)^2$. Finally, using equation (8) in this last equation gives:

(12) $(a - c)^2 = 8 - \frac{2}{e} - e^2$

Similarly, using equation (8) to rewrite equation (11), we get $b^2 + d^2 = 4 + e$. Therefore, $2b^2 + 2d^2 = 8 + 2e$, and hence from the algebraic identity $(b - d)^2 + (b + d)^2 = 2b^2 + 2d^2$, we get $(b - d)^2 = 2b^2 + 2d^2 - (b + d)^2 = 8 + 2e - (b + d)^2$. Finally, using equation (9) in this last equation gives $(b - d)^2 = 8 + 2e - \left(-\frac{1}{e}\right)^2$, and hence:

(13) $(b - d)^2 = 8 + 2e - \frac{1}{e^2}$

Now factor the left side of equation (5) and use the square roots of equations (12) and (13):

$$(a + c)(a - c) = b - d$$

$$e \bullet \sqrt{8 - \frac{2}{e} - e^2} = \pm\sqrt{8 + 2e - \frac{1}{e^2}}$$

This gives us an equation involving only e. We now proceed to solve this equation.

Square both sides of the last equation and divide through by e:

$$\frac{e^2\left(8 - \frac{2}{e} - e^2\right)}{e} = \frac{\left(8 + 2e - \frac{1}{e^2}\right)}{e}$$

$$8e - 2 - e^3 = \frac{8}{e} + 2 - \frac{1}{e^3}$$

$$e^3 - 8e + 4 + \frac{8}{e} - \frac{1}{e^3} = 0$$

(14) $\left(e^3 - \frac{1}{e^3}\right) - 8\left(e - \frac{1}{e}\right)e + 4 = 0$

Note that $e^3 - \frac{1}{e^3}$ is a difference of cubes that has $e - \frac{1}{e}$ as a factor. This observation possibly suggests the following variable change:

(15) $f = e - \frac{1}{e}$

It seems reasonable that if we are to rewrite equation (14) in terms of f, then the resulting equation will be cubic in f. This suggests that we express f^3 in terms of e and see what we get:

$$f^3 = (e)^3 + 3(e)^2\left(-\frac{1}{e}\right) + 3(e)\left(-\frac{1}{e}\right)^2 + \left(-\frac{1}{e}\right)^3$$
$$f^3 = e^3 - 3e + \frac{3}{e} - \frac{1}{e^3}$$
$$f^3 = e^3 - 3f - \frac{1}{e^3}$$

Rearranging the last equation gives:

$$f^3 + 3f = e^3 - \frac{1}{e^3}$$

The last few computations allow us to rewrite equation (14) in terms of f by replacing $e^3 - \frac{1}{e^3}$ with $f^3 + 3f$ and $e - \frac{1}{e}$ with f:

$$f^3 + 3f - 8f + 4 = 0$$
$$f^3 - 5f + 4 = 0$$

Since $f = 1$ is a zero of $f^3 - 5f + 4$, it follows that $f - 1$ is a factor of $f^3 - 5f + 4$. Thus, we can factor $f^3 - 5f + 4$ as follows:

(16) $(f - 1)\left(f^2 + f - 4\right) = 0$

Of course, it is immediately apparent by direct substitution that $f = 1$ is a zero of $f^3 - 5f + 4$. However, $f = 1$ can be discovered by the standard precalculus method for determining the rational solutions to an algebraic equation with integer coefficients.

BEGIN DIGRESSION

Chepmell makes the following comment at the end of his paper: *The factor* $(f - 1) = 0$ *is evidently connected with the equation* $x^{15} = 1$, *this being involved in our solution, as that* $x^{16} + x^{-16} = x + x^{-1}$ *is equally true in that case*. I believe that Chepmell is saying the $f = 1$ solution to equation (16) concerns the issue about extraneous solutions that I brought up at the beginning of this section. In fact, it is not difficult to see that every solution to $x^{15} = 1$ is also a solution to $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$. To show this, let p be a solution to $x^{15} = 1$. Then $p^{15} = 1$, from which it follows that $p^{16} = p$, and hence also $\frac{1}{p^{16}} = \frac{1}{p}$. Adding the last two equations gives $p^{16} + \frac{1}{p^{16}} = p + \frac{1}{p}$, which shows that p is a solution to $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$. Incidentally, for a trigonometric setting in which the real parts of the 15th roots of unity occur with the real parts of the 17th roots of unity, see Lebesgue [46].

Since $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$ is equivalent to $x^{32} - x^{17} - x^{15} + 1 = 0$, which is a polynomial equation of degree 32, the equation $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$ has 32 solutions counting multiplicity. Of these 32 solutions, 17 solutions will be the 17th roots of unity (since every solution to $x^{17} = 1$ is a solution to $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$) and 15 solutions will be the 15th roots of unity (since we just showed that every solution to $x^{15} = 1$ is a solution to $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$). We still seem to be missing a solution to

$x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$ since, with the exception of 1, which is both a 15th root of unity and a 17th root of unity, no 17th root of unity is a 15th root of unity (recall that the *n*th roots of unity are equally distributed, by distance, away from each other on the unit circle in the complex plane) and all these values for x give rise to only 17 + 15 – 1 = 31 different values for x (we subtract 1 because $x = 1$ satisfies both $x^{15} = 1$ and $x^{17} = 1$). To account for the 32nd value of x, we note that $x = 1$ is a double root of $x^{32} - x^{17} - x^{15} + 1 = 0$. A tedious way to show that $x = 1$ is a double root is to divide $x - 1$ into $x^{32} - x^{17} - x^{15} + 1$, using long division or synthetic division, and then show that $x = 1$ is a zero of the quotient. A quick way to show that $x = 1$ is a double root is to show that the derivative of $x^{32} - x^{17} - x^{15} + 1$ equals zero when $x = 1$. From these observations we get a complete account of the 32 solutions to $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$:

- 2 of the solutions correspond to $x = 1$ being a double root
- 14 of the solutions are the nonreal solutions to $x^{15} = 1$
- 16 of the solutions are the nonreal solutions to $x^{17} = 1$

Later we will see that the solutions to $x^{16} + \frac{1}{x^{16}} = x + \frac{1}{x}$ that arise from $f \neq 1$ include all of the 16 nonreal solutions to $x^{17} = 1$ (and no other values for x), and hence $f = 1$ must give rise to all 14 nonreal solutions to $x^{15} = 1$. We can also algebraically deduce the fact that none of the solutions for x that arise from $f = 1$ is a nonreal solution to $x^{17} = 1$ (and hence each solution that arises from $f = 1$ must be a nonreal solution to $x^{15} = 1$), but the only way I know to do this involves bringing in something that is just a little outside the realm of ordinary high school algebra. The *Eisenstein irreducibility criterion* implies that $\frac{x^{17} - 1}{x - 1} = x^{16} + x^{15} + \ldots + x^2 + x + 1$ is irreducible over the rationals, a result that can be found in many abstract algebra texts and older theory of equations texts: Hadlock [21] (Exercise 8 on p. 70, solution on pp. 252-253), Papantonopoulou [30] (pp. 255-256), Stewart [33] (pp. 40-42 and p. 217), and Klein [45] (end of Chapter III, pp. 21-23). Since all values for x that arise from $f = 1$ can be obtained by solving at most three quadratic equations (this is also true for the two other values of f, as we will see shortly), where the first equation has integer coefficients (use $f = 1$ in equation (17) below), it follows that each such value for x is a solution to an at most 8th degree polynomial with integer coefficients, a result that would contradict the Eisenstein irreducibility criterion if any of these values for x were zeros of the polynomial $\frac{x^{17} - 1}{x - 1}$.

END DIGRESSION

Excluding $f = 1$, the solutions to equation (16) are the solutions to $f^2 + f - 4 = 0$:

$$f_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{17}$$
$$f_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{17}$$

To summarize what follows, these 2 values for f give rise to 4 values for e, which in turn give rise to 8 values for a, which in turn give rise to all 16 nonreal solutions to $x^{17} = 1$.

To obtain the 4 values for e, first multiply both sides of equation (15) by e:

(17) $e^2 - fe - 1 = 0$

Applying the quadratic formula with $f_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{17}$ gives:

$$e_1 = -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 - 2\sqrt{17}}$$
$$e_2 = -\frac{1}{4} + \frac{1}{4}\sqrt{17} - \frac{1}{4}\sqrt{34 - 2\sqrt{17}}$$

Applying the quadratic formula with $f_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{17}$ gives:

$$e_3 = -\frac{1}{4} - \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 + 2\sqrt{17}}$$
$$e_4 = -\frac{1}{4} - \frac{1}{4}\sqrt{17} - \frac{1}{4}\sqrt{34 + 2\sqrt{17}}$$

Now that we have explicit expressions for the 4 values of e, we can obtain the explicit expressions for the 8 values of a as follows. Using equation (8), which is $e = a + c$, and the square root of equation (12), which is $a - c = \pm\sqrt{8 - \frac{2}{e} - e^2}$, we get:

$$(a + c) + (a - c) = e \pm \sqrt{8 - \frac{2}{e} - e^2}$$
$$2a = e \pm \sqrt{8 - \frac{2}{e} - e^2}$$

Therefore,

(18) $a = \frac{1}{2}e \pm \frac{1}{2}\sqrt{8 - \frac{2}{e} - e^2}$

Finally, from $a = x + \frac{1}{x}$ (definition of a, near the beginning), we can obtain explicit expressions for the 16 nonreal values of x by solving $x^2 - ax + 1 = 0$:

$$x = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

Remark: From equation (18) we see that each of the 4 values for e gives at most 2 values for a, so there are most 8 values for a. To actually show that the 8 values represented by the right side of equation (18) are distinct from each other, we would have to verify results such as $\frac{1}{2}e_1 + \frac{1}{2}\sqrt{8 - \frac{2}{e_1} - (e_1)^2} \neq \frac{1}{2}e_2 - \frac{1}{2}\sqrt{8 - \frac{2}{e_2} - (e_2)^2}$, where e_1 and e_2 are two (not necessarily distinct) values for e. A similar comment applies to the number of values of x. I do not know of an easy non-computational way to verify such results, however.

In Section 4 I will carry out the process entirely "by hand" for a value of x that allows us to obtain an explicit expression for $\cos\left(\frac{2\pi}{17}\right)$, and I will then show that it agrees with the explicit expression that Gauss got. However, before doing this, I thought it would be of interest to give approximate decimal values (truncated, not rounded) for all 8 values of a and all 16 values of x, along with their corresponding trigonometric forms. I resorted to computer/calculator calculations in order to match the decimal values with the trigonometric forms.

$$e_1 = -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 - 2\sqrt{17}}$$

$$a_1 = 1.8649...$$

$$x_1 = (0.9324...) + (0.3612...)i = \cos\left(\frac{2\pi}{17}\right) + i\sin\left(\frac{2\pi}{17}\right)$$

$$x_2 = (0.9324...) - (0.3612...)i = \cos\left(\frac{32\pi}{17}\right) + i\sin\left(\frac{32\pi}{17}\right)$$

$$a_2 = 0.1845...$$

$$x_3 = (0.0922...) + (0.9957...)i = \cos\left(\frac{8\pi}{17}\right) + i\sin\left(\frac{8\pi}{17}\right)$$

$$x_4 = (0.0922...) - (0.9957...)i = \cos\left(\frac{26\pi}{17}\right) + i\sin\left(\frac{26\pi}{17}\right)$$

$$e_2 = -\frac{1}{4} + \frac{1}{4}\sqrt{17} - \frac{1}{4}\sqrt{34 - 2\sqrt{17}}$$

$$a_3 = 1.4780...$$

$$x_5 = (0.7390...) + (0.6736...)i = \cos\left(\frac{4\pi}{17}\right) + i\sin\left(\frac{4\pi}{17}\right)$$
$$x_6 = (0.7390...) - (0.6736...)i = \cos\left(\frac{30\pi}{17}\right) + i\sin\left(\frac{30\pi}{17}\right)$$

$$a_4 = -1.9659...$$

$$x_7 = (-0.9829...) + (0.1837...)i = \cos\left(\frac{16\pi}{17}\right) + i\sin\left(\frac{16\pi}{17}\right)$$
$$x_8 = (-0.9829...) - (0.1837...)i = \cos\left(\frac{18\pi}{17}\right) + i\sin\left(\frac{18\pi}{17}\right)$$

$$e_3 = -\frac{1}{4} - \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 + 2\sqrt{17}}$$

$$a_5 = 0.8914...$$

$$x_9 = (0.4457...) + (0.8951...)i = \cos\left(\frac{6\pi}{17}\right) + i\sin\left(\frac{6\pi}{17}\right)$$
$$x_{10} = (0.4457...) - (0.8951...)i = \cos\left(\frac{28\pi}{17}\right) + i\sin\left(\frac{28\pi}{17}\right)$$

$$a_6 = -0.5473...$$

$$x_{11} = (-0.2736...) + (0.9618...)i = \cos\left(\frac{10\pi}{17}\right) + i\sin\left(\frac{10\pi}{17}\right)$$
$$x_{12} = (-0.2736...) - (0.9618...)i = \cos\left(\frac{24\pi}{17}\right) + i\sin\left(\frac{24\pi}{17}\right)$$

$$e_4 = -\frac{1}{4} - \frac{1}{4}\sqrt{17} - \frac{1}{4}\sqrt{34 + 2\sqrt{17}}$$

$$a_7 = -1.2052...$$

$$x_{13} = (-0.6026...) + (0.7980...)i = \cos\left(\frac{12\pi}{17}\right) + i\sin\left(\frac{12\pi}{17}\right)$$
$$x_{14} = (-0.6026...) - (0.7980...)i = \cos\left(\frac{22\pi}{17}\right) + i\sin\left(\frac{22\pi}{17}\right)$$

$a_8 = -1.7004\ldots$

$$x_{15} = (-0.8502\ldots) + (0.5264\ldots)i = \cos\left(\frac{14\pi}{17}\right) + i\sin\left(\frac{14\pi}{17}\right)$$
$$x_{16} = (-0.8502\ldots) - (0.5264\ldots)i = \cos\left(\frac{20\pi}{17}\right) + i\sin\left(\frac{20\pi}{17}\right)$$

Remark: Note that nonreal values did not occur until the last step, when the numerical values for x were obtained. At first I thought this was a nice coincidence (if imaginary values had shown up earlier, then we would be faced with solving quadratic equations with nonreal coefficients, and this would make the computations by hand quite a bit more tedious), but after some thought I realized this is not really all that surprising. It turns out to always be the case that, when obtaining the value of a constructible number, we can arrange that imaginary numbers do not appear until solving the final quadratic equation. This is a consequence of the fact (proved below) that every polynomial with real coefficients can be factored into linear and/or quadratic polynomials with real coefficients. More precisely, it follows from the fact that a polynomial, all of whose zeros are constructible, can be factored into linear and/or quadratic polynomials with coefficients that are both real and constructible. This more precise version will be apparent from the proof that follows, since this proof will show that the coefficients of the linear and quadratic factors arise from simple arithmetic computations applied to the original polynomial's zeros (thus, not taking us out of the realm of constructible numbers). [Proof of above "fact": The nonreal zeros of a polynomial with real coefficients occur in complex conjugate pairs, and each such pair $\alpha \pm \beta i$ is associated with the factor $(x - \alpha)^2 + \beta^2$. Thus, the polynomial (assumed, without loss of generality, to have a leading coefficient of 1) can be factored as the product of $(x - \alpha_1)^2 + \beta_1^2$, $(x - \alpha_2)^2 + \beta_2^2$, ... and $(x - r_1)$, $(x - r_2)$, ..., where the nonreal roots are $\alpha_1 \pm \beta_1 i$, $\alpha_2 \pm \beta_2 i$, ... and the real roots are r_1, r_2, ...]

4. Obtaining a Square Root Expression for $\cos\left(\frac{2\pi}{17}\right)$

Gauss gave the following expression for $\cos\left(\frac{2\pi}{17}\right)$ in Article 365 of his **Disquisitiones Arithmeticæ** (p. 662 of the original 1801 edition; p. 454 of the 1986 English translation):

$$-\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

In what follows I will show in great detail how one can obtain essentially this expression from the solution given in Section 3, and I will do this using only "paper and pencil" methods. In fact, I will almost completely by-pass even the need for computations that make use of grade school arithmetic algorithms. In doing this, I will endeavor as much as possible to demonstrate efficient computational strategies that were probably second-nature to students in the 1800s, but which have now largely atrophied into little known methods in our present calculator-dependent age.

From the values I gave earlier, $\cos\left(\frac{2\pi}{17}\right)$ is the real part of x_1, which happens to be $\frac{1}{2}a_1$. Thus, our strategy will be to obtain an explicit expression for the value of f_1, then an explicit expression for the value of e_1, and finally an explicit expression for the value of a_1. For notational simplicity, I will omit subscripts in what follows.

The first equation to solve arises from equation (16), and this equation is:

$$f^2 + f - 4 = 0$$

Using the quadratic formula, we get:

$$f = \frac{-1 \pm \sqrt{1^2 - 4(1)(-4)}}{2} = \frac{-1 \pm \sqrt{17}}{2}$$

The value we want corresponds to the plus sign:

$$f = -\frac{1}{2} + \frac{1}{2}\sqrt{17}$$

To obtain an explicit expression for the value of e, we solve equation (17):

$$e^2 - fe - 1 = 0$$

$$e^2 + \left(\frac{1}{2} - \frac{1}{2}\sqrt{17}\right)e - 1 = 0$$

Using the quadratic formula, we get:

$$e = \frac{-\left(\frac{1}{2} - \frac{1}{2}\sqrt{17}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{1}{2}\sqrt{17}\right)^2 - 4(1)(-1)}}{2}$$

The value we want corresponds to the plus sign:

$$\frac{-\frac{1}{2} + \frac{1}{2}\sqrt{17} + \sqrt{\left(\frac{1}{2}\right)^2 (1 - \sqrt{17})^2 + 4}}{2}$$

$$= -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{2}\sqrt{\left(\frac{1}{4}\right)(1 - \sqrt{17})^2 + \frac{16}{4}}$$

$$= -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sqrt{(1 - \sqrt{17})^2 + 16}$$

$$= -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{1 - 2\sqrt{17} + 17 + 16}$$

Thus, we get:

(19) $e = -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 - 2\sqrt{17}}$

To obtain an explicit expression for the value of a, we make use of equation (18), choosing the plus sign:

$$a = \frac{1}{2}e + \frac{1}{2}\sqrt{8 - \frac{2}{e} - e^2}$$

Almost all of our remaining work will be involved in putting this expression for the value of a into "simple form". Having done this, we get:

$$\cos\left(\frac{2\pi}{17}\right) = \frac{1}{2}a = \frac{1}{4}e + \frac{1}{4}\sqrt{8 - \frac{2}{e} - e^2}$$

To simplify the above expression for the value of a, it will be helpful to first obtain simplified expressions for $-e^2$ and $-\frac{2}{e}$.

Expanding $-e^2$ involves squaring a trinomial, which can be done by adding the squares of each term to twice the sum of the three possible products of two distinct terms, as shown by the following algebraic identity:

$$(k + m + n)^2 = k^2 + m^2 + n^2 + 2km + 2kn + 2mn$$

Carrying this out gives:

$$\left(-\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 - 2\sqrt{17}}\right)^2$$

$$= \left(\frac{1}{4}\right)^2\left(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right)^2$$

$$= \frac{1}{16}\left[(-1)^2 + \left(\sqrt{17}\right)^2 + \left(\sqrt{34 - 2\sqrt{17}}\right)^2\right] +$$
$$\frac{1}{16}\left[2(-1)\left(\sqrt{17}\right) + 2(-1)\left(\sqrt{34 - 2\sqrt{17}}\right) + 2\left(\sqrt{17}\right)\left(\sqrt{34 - 2\sqrt{17}}\right)\right]$$

$$= \frac{1}{16}\left[1 + 17 + 34 - 2\sqrt{17} - 2\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}} + 2\sqrt{17\left(34 - 2\sqrt{17}\right)}\right]$$

$$= \frac{1}{16}\left[52 - 4\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}} + 2\sqrt{578 - 34\sqrt{17}}\right]$$

$$= \frac{13}{4} - \frac{1}{4}\sqrt{17} - \frac{1}{8}\sqrt{34 - 2\sqrt{17}} + \frac{1}{8}\sqrt{578 - 34\sqrt{17}}$$

Therefore, we have:

(20) $-e^2 = -\frac{13}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{8}\sqrt{34 - 2\sqrt{17}} - \frac{1}{8}\sqrt{578 - 34\sqrt{17}}$

Now we turn our attention to $-\frac{2}{e}$:

$$-\frac{2}{e} = \frac{-2}{-\frac{1}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{4}\sqrt{34 - 2\sqrt{17}}}$$

(21) $-\frac{2}{e} = \frac{8}{1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}}}$

In order to obtain an expanded expression for the value of $-\frac{2}{e}$ that is similar to the expanded expression we obtained for the value of $-e^2$, we will rationalize the denominator of the right side of equation (21). One way to rationalize a trinomial denominator is to multiply both the numerator and denominator by three appropriate conjugates, using the fact that

$$\begin{aligned}
&(A+B+C)(A+B-C)(A-B+C)(A-B-C)\\
&= \left[(A+B)^2 - C^2\right]\left[(A-B)^2 - C^2\right]\\
&= (A+B)^2 (A-B)^2 - C^2 (A+B)^2 - C^2 (A-B)^2 + C^4\\
&= \left(A^2 - B^2\right)^2 - C^2\left[A^2 + 2AB + B^2 + A^2 - 2AB + B^2\right] + C^4\\
&= \left(A^2 - B^2\right)^2 - C^2\left(2A^2 + 2B^2\right) + C^4
\end{aligned}$$

involves only even powers of A, B, and C. I see no compelling reason to expand this expression further, so I will use the final version above in carrying out the radical computations.

To see the big picture, note that if we have a trinomial denominator of the $(-\,-)$ type, then we would multiply it by trinomials of the $(+\,+)$, $(+\,-)$, and $(-\,+)$ types. Similarly, if we had a trinomial of the $(-\,+)$ type, then we would multiply it by trinomials of the $(+\,+)$, $(+\,-)$, and $(-\,-)$ types. Note the analogy with binomial denominators, where a $(+)$ type (for example, $2 + \sqrt{3}$ or $4\sqrt{5} + \sqrt{6}$) is multiplied by a $(-)$ type ($2 - \sqrt{3}$ and $4\sqrt{5} - \sqrt{6}$ for the examples I just gave) and a $(-)$ type is multiplied by a $(+)$ type. For more about this method, see the last half of <**http://tinyurl.com/4grva4w**>. The method I describe there, in which roots of unity are used to rationalize sums and differences of cube roots, quartic roots, etc. (this reduces to using plus and minus signs in the case of sums and differences of square roots), is possibly due to Joseph Louis Lagrange from about 1767. (This speculation on my part is based on p. 691 in Volume II of Dickson's **History of the Theory of Numbers**.)

From the above identity it follows that multiplying $1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}}$ by

$$\left(1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right)\left(1 + \sqrt{17} - \sqrt{34 - 2\sqrt{17}}\right)\left(1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right)$$

will at least remove all the outer radicals. That is, our strategy will be to multiply both numerator and denominator of the right side of equation (21) by $(+\,+)$, $(+\,-)$, and $(-\,+)$ expressions that are based on the $(-\,-)$ expression in the denominator.

We will deal with the denominator first. The new denominator will be

$$\left(A^2 - B^2\right)^2 - C^2\left(2A^2 + 2B^2\right) + C^4,$$

where $A = 1$, $B = \sqrt{17}$, and $C = \sqrt{34 - 2\sqrt{17}}$.

Taking this one step at a time, we get:

$$\left(A^2 - B^2\right)^2 = (1 - 17)^2$$
$$= 16^2$$

$$-C^2\left(2A^2 + 2B^2\right)$$
$$= -\left(34 - 2\sqrt{17}\right)\left(2\bullet 1^2 + 2\bullet 17\right) = -(2)\left(17 - \sqrt{17}\right)(36) = -72\left(17 - \sqrt{17}\right)$$
$$= (-72)(17) + 72\sqrt{17}$$

$$C^4 = \left(34 - 2\sqrt{17}\right)^2 = \left[2\left(17 - \sqrt{17}\right)\right]^2 = \left[2\left(\sqrt{17}\right)\left(\sqrt{17} - 1\right)\right]^2$$
$$= (4)(17)\left(17 - 2\sqrt{17} + 1\right) = (4)(17)\left(18 - 2\sqrt{17}\right) = (8)(17)\left(9 - \sqrt{17}\right)$$
$$= (72)(17) - (8)(17)\sqrt{17}$$

Adding these results, we get:

$$\left(A^2 - B^2\right)^2 - C^2\left(2A^2 + 2B^2\right) + C^4$$
$$= 16^2 + (-72)(17) + 72\sqrt{17} + (72)(17) - (8)(17)\sqrt{17}$$
$$= 16^2 + 72\sqrt{17} - (8)(17)\sqrt{17}$$
$$= 16^2 + 8\left(9\sqrt{17} - 17\sqrt{17}\right)$$
$$= 16^2 + 8\left(-8\sqrt{17}\right)$$
$$= 16^2 - 8^2\sqrt{17}$$

Now we will deal with the numerator. The new numerator will be

$$8(A + B + C)(A + B - C)(A - B + C)$$
$$= 8\left[(A + B)^2 - C^2\right](A - B + C),$$

where $A = 1$, $B = \sqrt{17}$, and $C = \sqrt{34 - 2\sqrt{17}}$.

Again, we take this one step at a time:

$$(A + B)^2 - C^2$$
$$= \left(1 + \sqrt{17}\right)^2 - \left(34 - 2\sqrt{17}\right)$$
$$= 1 + 2\sqrt{17} + 17 - 34 + 2\sqrt{17}$$
$$= -16 + 4\sqrt{17}$$
$$= 4\left(-4 + \sqrt{17}\right)$$

$A - B + C$
$= 1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}$

Therefore, the new numerator is equal to:

$$
\begin{aligned}
&8\left[(A + B)^2 - C^2\right](A - B + C)\\
&= 8\left[4\left(-4 + \sqrt{17}\right)\right]\left(1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right)\\
&= 32\left(-4 + \sqrt{17}\right)\left(1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right)\\
&= 32\left(-4 + 4\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17} - 17 + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)\\
&= 32\left(-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)
\end{aligned}
$$

Putting the new denominator and new numerator together, we get:

$$-\frac{2}{e} = \frac{8}{1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}}}$$

$$= \frac{32\left(-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)}{16^2 - 8^2\sqrt{17}}$$

$$= \frac{32\left(-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)}{(32)(8) - (32)(2)\sqrt{17}}$$

$$= \frac{-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}}{8 - 2\sqrt{17}}$$

The denominator is not yet rationalized, but it is clear what we can do, namely multiply both the numerator and the denominator by $8 + 2\sqrt{17}$:

$$\frac{\left(-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)\left(8 + 2\sqrt{17}\right)}{\left(8 - 2\sqrt{17}\right)\left(8 + 2\sqrt{17}\right)}$$

$$= \frac{\left(-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)(2)\left(4 + \sqrt{17}\right)}{64 - (4)(17)}$$

$$= \frac{\left(-21 + 5\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} + \sqrt{17}\sqrt{34 - 2\sqrt{17}}\right)\left(4 + \sqrt{17}\right)}{32 - (2)(17)}$$

The above denominator is $32 - (2)(17) = -2$. The above numerator initially seems rather messy, but it cleans up quite nicely:

$$-84 + 20\sqrt{17} - 16\sqrt{34 - 2\sqrt{17}} + 4\sqrt{17}\sqrt{34 - 2\sqrt{17}}$$
$$-21\sqrt{17} + (5)(17) - 4\sqrt{17}\sqrt{34 - 2\sqrt{17}} + 17\sqrt{34 - 2\sqrt{17}}$$

$$= 1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}$$

Therefore, we get:

$$-\frac{2}{e} = \frac{8}{1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}}} = \frac{1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}}{-2}$$

(22) $-\frac{2}{e} = -\frac{1}{2} + \frac{1}{2}\sqrt{17} - \frac{1}{2}\sqrt{34 - 2\sqrt{17}}$

To review, we are seeking an expression for

$$\cos\left(\frac{2\pi}{17}\right) = \frac{1}{2}a = \frac{1}{4}e + \frac{1}{4}\sqrt{8 - \frac{2}{e} - e^2}$$

and from equations (19), (20), and (22) we have the following:

$$\frac{1}{4}e = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}}$$
$$-\frac{2}{e} = -\frac{1}{2} + \frac{1}{2}\sqrt{17} - \frac{1}{2}\sqrt{34 - 2\sqrt{17}}$$
$$-e^2 = -\frac{13}{4} + \frac{1}{4}\sqrt{17} + \frac{1}{8}\sqrt{34 - 2\sqrt{17}} - \frac{1}{8}\sqrt{578 - 34\sqrt{17}}$$

Hence:

$$-\frac{2}{e} - e^2 = -\frac{15}{4} + \frac{3}{4}\sqrt{17} - \frac{3}{8}\sqrt{34 - 2\sqrt{17}} - \frac{1}{8}\sqrt{578 - 34\sqrt{17}}$$

$$8 - \frac{2}{e} - e^2 = \frac{17}{4} + \frac{3}{4}\sqrt{17} - \frac{3}{8}\sqrt{34 - 2\sqrt{17}} - \frac{1}{8}\sqrt{578 - 34\sqrt{17}}$$

$$\frac{1}{4}\sqrt{8 - \frac{2}{e} - e^2} = \frac{1}{4}\sqrt{\frac{17}{4} + \frac{3}{4}\sqrt{17} - \frac{3}{8}\sqrt{34 - 2\sqrt{17}} - \frac{1}{8}\sqrt{578 - 34\sqrt{17}}}$$
$$= \frac{1}{4}\sqrt{\left(\frac{1}{16}\right)\left(68 + 12\sqrt{17} - 6\sqrt{34 - 2\sqrt{17}} - 2\sqrt{578 - 34\sqrt{17}}\right)}$$
$$= \frac{1}{16}\sqrt{68 + 12\sqrt{17} - 6\sqrt{34 - 2\sqrt{17}} - 2\sqrt{578 - 34\sqrt{17}}}$$

$$\cos\left(\frac{2\pi}{17}\right) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}}$$
$$+ \frac{1}{16}\sqrt{68 + 12\sqrt{17} - 6\sqrt{34 - 2\sqrt{17}} - 2\sqrt{578 - 34\sqrt{17}}}$$

It is possible to further compress the 3-nested radical by making more use of the numerical appearances of 17:

$$\sqrt{(4)(17) + 12\sqrt{17} - 6\sqrt{(2)(17) - 2\sqrt{17}} - 2\sqrt{(2)\left(17^2\right) - (2)(17)\sqrt{17}}}$$

To make this more transparent, we replace $\sqrt{17}$ with u (and hence 17 and 17^2 with u^2 and u^4):

$$\sqrt{4u^2 + 12u - 6\sqrt{2u^2 - 2u} - 2\sqrt{2u^4 - 2u^2 \bullet u}}$$

$$= \sqrt{4u(u + 3) - 6\sqrt{2u^2 - 2u} - 2u\sqrt{2u^2 - 2u}}$$

$$= \sqrt{4u(u + 3) - (6 + 2u)\sqrt{2u^2 - 2u}}$$

$$= \sqrt{4u(u + 3) - 2(u + 3)\sqrt{2u^2 - 2u}}$$

$$= \sqrt{(u + 3)\left(4u - 2\sqrt{2u^2 - 2u}\right)}$$

Therefore, the 3-nested radical can be expressed as

$$\sqrt{\left(3 + \sqrt{17}\right)\left(4\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}}\right)},$$

and hence we get:

$$\cos\left(\frac{2\pi}{17}\right) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{16}\sqrt{\left(3 + \sqrt{17}\right)\left(4\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}}\right)}$$

We now compare the value of the expression we got with the value of the expression Gauss gave, which is

$$\cos\left(\frac{2\pi}{17}\right) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

Looking at both expressions shows that we need to compare the value of

$$\frac{1}{16}\sqrt{68 + 12\sqrt{17} - 6\sqrt{34 - 2\sqrt{17}} - 2\sqrt{578 - 34\sqrt{17}}}$$

with the value of

$$\frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$
$$= \frac{1}{16}\sqrt{68 + 12\sqrt{17} - 4\sqrt{34 - 2\sqrt{17}} - 8\sqrt{34 + 2\sqrt{17}}}$$

This reduces to verifying the following equalities:

$$-6\sqrt{34 - 2\sqrt{17}} - 2\sqrt{578 - 34\sqrt{17}} = -4\sqrt{34 - 2\sqrt{17}} - 8\sqrt{34 + 2\sqrt{17}}$$

$$-2\sqrt{578 - 34\sqrt{17}} = 2\sqrt{34 - 2\sqrt{17}} - 8\sqrt{34 + 2\sqrt{17}}$$

$$-\sqrt{(2)\left(17^2\right) - 34\sqrt{17}} = \sqrt{34 - 2\sqrt{17}} - 4\sqrt{34 + 2\sqrt{17}}$$

Since both sides are clearly negative (in the case of the right side, $\sqrt{34 - 2\sqrt{17}}$ is less than $\sqrt{34 + 2\sqrt{17}}$, and hence $\sqrt{34 - 2\sqrt{17}}$ is certainly less than $4\sqrt{34 + 2\sqrt{17}}$), squaring both sides is a reversible operation. I leave the details of squaring both sides to the reader. The computations are neither lengthy nor difficult, at least not when carried out so as to exploit all the appearances of 17 that show up. As an example of a calculation that showed up near the beginning when I did this, $\sqrt{34^2 - (4)(17)}$ is equal to $\sqrt{\left(2^2\right)\left(17^2\right) - (4)(17)} = \sqrt{(4)(17)(17 - 1)} = (2)(4)\sqrt{17} = 8\sqrt{17}$.

5. The Gauss Method for Solving $x^{17} = 1$

In this section I will outline how Gauss solved $x^{17} = 1$. However, I will not discuss the theoretical framework that Gauss developed in order to solve $x^{17} = 1$, a framework that can be applied to $x^n = 1$ for an arbitrary positive integer n. I should point out, by the way, that in the case of an arbitrary positive integer n the phrase "solve" is a bit imprecise. What can be done for each solution is to show how to reach that solution by solving a sequence of equations having a certain kind of minimal complexity, such as all the equations are at most quadratic or all the equations are at most cubic. This general framework that Gauss developed is extensively discussed in many of the references listed later. In particular, Hadlock [21] (pp. 113-118) gives an especially detailed treatment in which the various 17th roots of unity are identified, by algebraic methods, with the various sign choices made in solving the quadratic equations. That is, for the various quadratic equations that arise, Hadlock identifies the specific sign choices (the $\pm$ that shows up when using the quadratic formula) one would make in order to reach a specified solution to $x^{17} = 1$, such as $x = \cos\left(\frac{24\pi}{17}\right) + i\sin\left(\frac{24\pi}{17}\right)$.

Since $x = 1$ is a zero of $x^{17} - 1$, it follows that $x - 1$ is a factor of $x^{17} - 1$:

$$(x-1)\left(x^{16} + x^{15} + x^{14} + \ldots + x^2 + x + 1\right) = 0$$

Thus, we are led to solve:

$$x^{16} + x^{15} + x^{14} + \ldots + x^2 + x + 1 = 0$$

Now let

$$A_1 = x + x^2 + x^4 + x^8 + x^9 + x^{13} + x^{15} + x^{16}$$
$$A_2 = x^3 + x^5 + x^6 + x^7 + x^{10} + x^{11} + x^{12} + x^{14}$$

An obvious question is why we would choose to use (or even know to use) this particular partition of the exponents 1, 2, 3, ..., 14, 15, 16 into these two collections rather than into some other two collections, such as 1, 2, ..., 7, 8 and 9, 10, ..., 15, 16. Indeed, the choice of what partition to use is not arbitrary, and it arises from the following considerations. Start with the sequence of consecutive powers of 3 from 3^0 through 3^{15}. (Why 3? This is an example of where I am skipping the theoretical basis behind the Gauss method.)

$$3^0,\ 3^1,\ 3^2,\ 3^3,\ 3^4,\ \ldots,\ 3^{13},\ 3^{14},\ 3^{15}$$

$$1,\ 3,\ 9,\ 27,\ 81,\ \ldots,\ 1594323,\ 4782969,\ 14348907$$

Next, divide each of these powers of 3 by 17, express the results in quotient plus remainder form, and only keep the remainders:

$1 = (0)(17) + 1 \quad \rightarrow \quad 1$
$3 = (0)(17) + 3 \quad \rightarrow \quad 3$
$9 = (0)(17) + 9 \quad \rightarrow \quad 9$
$27 = (1)(17) + 10 \quad \rightarrow \quad 10$
$81 = (4)(17) + 13 \quad \rightarrow \quad 13$
...
$1594323 = (93783)(17) + 12 \quad \rightarrow \quad 12$
$4782969 = (281351)(17) + 2 \quad \rightarrow \quad 2$
$14348907 = (844053)(17) + 6 \quad \rightarrow \quad 6$

We will use this new ordered list of the first 16 positive integers to obtain A_1 and A_2. In the case of A_1, we take the 1st, 3rd, ..., 13th, 15th entries in this new list (i.e. 1, 9, 13, ..., 4, 2) and use the corresponding powers of x. In the case of A_2, we take the 2nd, 4th, ..., 14th, 16th entries in this new list (i.e. 3, 10, 5, ..., 12, 6) and use the corresponding powers of x for A_2.

I now claim that $A_1 + A_2 = -1$ and $A_1 A_2 = -4$. It is easy to show that $A_1 + A_2 = -1$, since $A_1 + A_2$ is equal to $x + x^2 + x^3 + \ldots + x^{15} + x^{16}$ and we know that $1 + \left(x + x^2 + x^3 + \ldots + x^{15} + x^{16}\right) = 0$.

It is a little more tedious to show that $A_1 A_2 = -4$. First, expand $A_1 A_2$ by expanding xA_2 (8 terms), expanding $x^2 A_2$ (8 terms), expanding $x^4 A_2$ (8 terms), ..., and expanding $x^{16} A_2$ (8 terms), for a total of 64 terms. Next, rewrite these terms so that only exponents from 1 through 16 are used, which is possible by making use of the fact that $x^{17} = 1$ (and hence, $x^{18} = x^{17}x = x$, $x^{19} = x^{17}x^2 = x^2$, $x^{20} = x^{17}x^3 = x^3$, etc.). Finally, by carefully examining these 64 terms, you will find that each of the 16 terms in $x + x^2 + x^3 + \ldots + x^{15} + x^{16}$ appears exactly 4 times, from which we conclude that $A_1 A_2 = 4\left(x + x^2 + x^3 + \ldots + x^{15} + x^{16}\right) = 4(A_1 + A_2) = 4(-1) = -4$.

The equations $A_1 + A_2 = -1$ and $A_1 A_2 = -4$ can be solved for A_1 and A_2 by substitution:

From $A_2 = -1 - A_1$, we get $A_1(-1 - A_1) = -4$. This is a quadratic equation for A_1, which we can solve to get two values for A_1. Using these values for A_1, we can obtain the corresponding values for $A_2 = -1 - A_1$. This gives us two values for the pair (A_1, A_2).

We now define B_1 and B_2 by using the 8 terms that made up our definition of A_1:

$$B_1 = x + x^4 + x^{13} + x^{16}$$
$$B_2 = x^2 + x^8 + x^9 + x^{15}$$

As one might expect, an arbitrary partition of the terms in A_1 into two groups of 4 terms each may not work, so I will explain the method by which the exponents appearing in A_1 are partitioned. Recall that when we used the remainders obtained by dividing consecutive powers of 3 by 17, the exponents we used for A_1 were the 1st, 3rd, ..., 13th, 15th values in the original ordering of the remainders. In the case of B_1, the exponents we use are those that occur as every 4th value in the original list of remainders, beginning with the 1st. Similarly, in the case of B_2, the exponents we use are those that occur as every 4th value in the original list of remainders, beginning with the 3rd. Thus, the terms we are using for B_1 and B_2 are chosen to be the odd and even numbered terms in the list of terms making up A_1, when the terms in A_1 are listed in their original ordering. Of course, there is a reason behind this repeated even/odd alteration in picking the powers of x (first for A_1 and A_2, and now for B_1 and B_2), but this is another example of where I am skipping the theoretical basis behind the Gauss method.

Since $B_1 + B_2 = A_1$ and the values for A_1 are known, it follows that the values for $B_1 + B_2$ are known. Moreover, in the same way that we previously showed $A_1 A_2 = -4$, we can show (less tediously this time) that $B_1 B_2 = -1$. Hence, both $B_1 + B_2$ and $B_1 B_2$ have known values. Therefore, the values for B_1 and B_2 can be found by solving a quadratic equation in the same way that we can find values for A_1 and A_2 by solving a quadratic equation.

Similar to how we defined B_1 and B_2, we define B_3 and B_4 by using the 8 terms whose sum is A_2:

$$B_3 = x^3 + x^5 + x^{12} + x^{14}$$
$$B_4 = x^6 + x^7 + x^{10} + x^{11}$$

Since $B_3 + B_4$ is equal to A_2, whose values are known, and $B_3 B_4$ is equal to -1, it follows that the values for B_3 and B_4 can be found by solving a quadratic equation.

Next, let

$$C_1 = x + x^{16}$$
$$C_2 = x^4 + x^{13}$$

Then $C_1 + C_2$ is equal to B_1, whose values are known, and it is easy to show that C_1C_2 is equal to B_3, whose values are known. Therefore, the values for C_1 and C_2 can be found by solving a quadratic equation.

Finally, we have $x + x^{16} = C_1$, where the values for C_1 are known. This last equation is a quadratic equation in disguise: Since $x^{17} = 1$, we have $x^{16} = \frac{1}{x}$, and hence $x + x^{16} = C_1$ becomes $x + \frac{1}{x} = C_1$, or $x^2 - C_1x + 1 = 0$.

6. Solving $x^{257} = 1$ by Quadratic Equations

Bishop [53] gave a specific sequence of 25 quadratic equations that can be used to obtain the value of $\cos\left(\frac{2\pi}{257}\right) + i\sin\left(\frac{2\pi}{257}\right)$, and these equations are listed below. Since Bishop's notation for the coefficients of the quadratic equations was chosen in accordance with the method he used to obtain the equations, his notation is less convenient than it could have been for simply listing equations. In what follows, I will use P_k and N_k to denote the solutions to the k th quadratic equation where, in using the quadratic formula, P_k corresponds to the positive sign and N_k corresponds to the negative sign. That is, if the k th equation is $ax^2 + bx + c = 0$, then $P_k = \frac{-b + \sqrt{b^2 - 4ac}}{2}$ and $N_k = \frac{-b - \sqrt{b^2 - 4ac}}{2}$. I am also including carefully calculated numerical values (truncated, not rounded) for each P_k and N_k.

EQUATION 1 $x^2 + x - 64 = 0$

$P_1 = 7.5156\ldots$
$N_1 = -8.5156\ldots$

EQUATION 2 $x^2 - P_1x + 16(P_1 + N_1) = 0$

$P_2 = 9.2460\ldots$
$N_2 = -1.7304\ldots$

EQUATION 3 $x^2 - N_1x + 16(P_1 + N_1) = 0$

$P_3 = 1.5841\ldots$
$N_3 = -10.0997\ldots$

EQUATION 4 $x^2 - P_2x + (2P_2 + 4N_2 + 5P_3 + 5N_3) = 0$

$P_4 = 11.8604\ldots$
$N_4 = -2.6143\ldots$

EQUATION 5 $x^2 - P_3x + (5P_2 + 5N_2 + 2P_3 + 4N_3) = 0$

$P_5 = 1.3214\ldots$
$N_5 = 0.2627\ldots$

EQUATION 6 $x^2 - N_2x + (4P_2 + 2N_2 + 5P_3 + 5N_3) = 0$

$P_6 = 2.2657\ldots$
$N_6 = -3.9962\ldots$

EQUATION 7 $x^2 - N_3x + (5P_2 + 5N_2 + 4P_3 + 2N_3) = 0$

$P_7 = -3.7133\ldots$
$N_7 = -6.3864\ldots$

EQUATION 8 $x^2 - P_4x + (2P_4 + N_4 + 2P_5 + 2P_6 + N_6) = 0$

$P_8 = 9.2291\ldots$
$N_8 = 2.6313\ldots$

EQUATION 9 $x^2 - N_5x + (P_5 + 2N_5 + 2N_6 + +P_7 + 2N_7) = 0$

$P_9 = 4.8904\ldots$
$N_9 = -4.6277\ldots$

EQUATION 10 $x^2 - P_6x + (P_4 + 2N_4 + 2P_6 + N_6 + 2P_7) = 0$

$P_{10} = 2.3751\ldots$
$N_{10} = -0.1093\ldots$

EQUATION 11 $x^2 - N_7 x + (2P_4 + 2P_5 + N_5 + P_7 + 2N_7) = 0$

$P_{11} = -2.9559\ldots$
$N_{11} = -3.4304\ldots$

EQUATION 12 $x^2 - N_4 x + (P_4 + 2N_4 + 2N_5 + P_6 + 2N_6) = 0$

$P_{12} = -0.7797\ldots$
$N_{12} = -1.8346\ldots$

EQUATION 13 $x^2 - P_5 x + (2P_5 + N_5 + 2P_6 + 2P_7 + N_7) = 0$

$P_{13} = 3.2707\ldots$
$N_{13} = -1.9493\ldots$

EQUATION 14 $x^2 - N_6 x + (2P_4 + N_4 + P_6 + 2N_6 + 2N_7) = 0$

$P_{14} = -0.8210\ldots$
$N_{14} = -3.1752\ldots$

EQUATION 15 $x^2 - P_7 x + (2N_4 + P_5 + 2N_5 + 2P_7 + N_7) = 0$

$P_{15} = 2.6866\ldots$
$N_{15} = -6.4000\ldots$

EQUATION 16 $x^2 - P_8 x + (P_8 + P_9 + P_{10} + P_{13}) = 0$

$P_{16} = 5.8509\ldots$
$N_{16} = 3.3781\ldots$

EQUATION 17 $x^2 - P_9 x + (P_9 + P_{10} + P_{11} + N_{14}) = 0$

$P_{17} = 4.6463\ldots$
$N_{17} = 0.2441\ldots$

EQUATION 18 $x^2 - P_{15} x + (N_8 + N_9 + N_{12} + P_{15}) = 0$

$P_{18} = 3.0605\ldots$
$N_{18} = -0.3739\ldots$

EQUATION 19 $x^2 - N_8 x + (N_8 + N_9 + N_{10} + N_{13}) = 0$

$P_{19} = 3.7210\ldots$
$N_{19} = -1.0897\ldots$

EQUATION 20 $x^2 - N_9 x + (N_9 + N_{10} + N_{11} + P_{14}) = 0$

$P_{20} = 1.4732\ldots$
$N_{20} = -6.1010\ldots$

EQUATION 21 $x^2 - N_{15} x + (P_8 + P_9 + P_{12} + N_{15}) = 0$

$P_{21} = -1.3833\ldots$
$N_{21} = -5.0167\ldots$

EQUATION 22 $x^2 - P_{16} x + (P_{17} + P_{18}) = 0$

$P_{22} = 3.8483\ldots$
$N_{22} = 2.0026\ldots$

EQUATION 23 $x^2 - P_{19}x + (P_{20} + P_{21}) = 0$

$P_{23} = 3.6967\ldots$
$N_{23} = 0.0243\ldots$

EQUATION 24 $x^2 - P_{22}x + P_{23} = 0$

$P_{24} = 1.9994\ldots$
$N_{24} = 1.8489\ldots$

EQUATION 25 $x^2 - P_{24}x + 1 = 0$

$P_{25} = (0.9997\ldots) + (0.0244\ldots)\mathrm{i} = \cos\left(\frac{2\pi}{257}\right) + \mathrm{i}\sin\left(\frac{2\pi}{257}\right)$
$N_{25} = (0.9997\ldots) - (0.0244\ldots)\mathrm{i} = \cos\left(\frac{2\pi}{257}\right) - \mathrm{i}\sin\left(\frac{2\pi}{257}\right)$

7. References: General and Historical

1 Raymond Clare Archibald (1875-1955), *Remarks on Klein's "Famous Problems of Elementary Geometry"*, American Mathematical Monthly **21** #8 (October 1914), 247-259.
This paper contains a lot of carefully considered historical comments.
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4 Samuel Barnard (??-??) and James Mark Child (1871-1960), **Higher Algebra**, MacMillan and Company, 1936, xiv + 585 pages.
See Chapter XI: *Reciprocal and Binomial Equations* (pp. 168-178), especially Exercise 22 on p. 178 in which the reader is asked to show that $\sin\left(\frac{2\pi}{17}\right)$ and $\cos\left(\frac{\pi}{17}\right)$ are equal to two square root expressions that are given in the statement of the exercise. The Gauss method for solving $x^{17} = 1$ is sketched in Section XI.5 on pp. 173-175.
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9 Florian Cajori (1859-1930), *Pierre Laurent Wantzel*, Bulletin of the American Mathematical Society **24** #7 (April 1918), 339-347.
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10 Horatio Scott Carslaw (1870-1954), *Gauss's theorem on the regular polygons which can be constructed by Euclid's method*, Proceedings of the Edinburgh Mathematical Society **28** (1910), 121-128.
In the first 3 pages Carslaw outlines a proof (some needed facts about irreducibility of polynomials are used without proof, although references are given) of which positive integers n it is the case that not all solutions to $x^n = 1$ are constructible. This is a result that Gauss stated without proof, although Carslaw attributes the result to Gauss. The remainder of Carslaw's paper consists of a detailed proof that, for the remaining positive integers n, all solutions to $x^n = 1$ are constructible.
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Gauss proved the following in his 1801 book **Disquisitiones Arithmeticae**: If n is a non-negative power of 2 times a product of distinct Fermat primes (this is intended to include the case of no Fermat primes for $n \geq 2$ and the case of exactly one Fermat prime), then each solution to $x^n = 1$ is a constructible number. In this book Gauss also stated the converse, namely if n is not of this form, then at least one solution to $x^n = 1$ is not constructible. However, Gauss never published a proof of this converse, nor has an outline of a proof been found in his unpublished manuscripts or correspondence. Clements discusses the different ways that various authors have or have not awarded credit to Gauss for the converse. This matter is also the topic of Kazarinoff [22].

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18 Leonard Eugene Dickson (1874-1954), **New First Course in the Theory of Equations**, John Wiley and Sons, 1939, ix + 185 pages.
See Chapter IV: *Impossibility of the Trisection of an Angle or Construction of Regular Polygons of Seven and Nine Sides by Ruler and Compasses* (pp. 30-41) and Chapter XII: *Roots of Unity and Regular Polygons* (pp. 163-175). The Gauss method for solving $x^{17} = 1$ is sketched in Article 107: *Regular Polygon of Seventeen Sides* (pp. 170-172).

19 Dewey Colfax Duncan (1898-1988), *A criticism of the treatment of the regular polygon constructions in certain well-known geometry texts*, School Science and Mathematics **34** #1 (January 1934), 50-57.
Duncan discusses various incorrect and misleading statements about the constructability of n-gons and Gauss's results that he found by surveying over 20 U.S. high school geometry texts published in the 1920s and early 1930s. In addition, Duncan gives an English translation of Articles 365-366 in Gauss's 1801 **Disquisitiones Arithmeticae**. (In Duncan's translation, the largest radical term in the expression for $\cos\left(\frac{2\pi}{17}\right)$ has a typo—one of the inner radical symbols extends too far.)

20 Russell Nelson Euler (1950-), *Reciprocal equations*, The Pentagon **53** #2 (Spring 1994), 34-39.
This is an elementary introduction (only high school algebra and precalculus concepts are used) consisting mainly of how to recognize and solve reciprocal equations. Example 5 (pp. 37-38) gives a detailed solution to $x^5 = 1$.
http://www.pentagon.kappamuepsilon.org/pentagon/Vol_53_Num_2_Spring_1994.pdf

21 Charles Robert Hadlock (1947-), **Field Theory and its Classical Problems**, Carus Mathematical Monographs #19, Mathematical Association of America, 1978, xvi + 323 pages.
Sections 1.1-1.5 (pp. 9-33) give a nice discussion, at a relatively elementary level, of issues relating to constructible numbers. See also Chapter 2 (pp. 59-121), especially Section 2.7: *Constructability of Regular Polygons II* (pp. 104-121). Section 2.7 gives an especially detailed treatment of the Gauss method for solving $x^{17} = 1$.

22 Nicholas D. Kazarinoff (1929-1991), *On who first proved the impossibility of constructing certain regular polygons with ruler and compass alone*, American Mathematical Monthly **75** #6 (June-July 1968), 647.
See also Clements [11] and Pierpont [31].

23 Nicholas D. Kazarinoff (1929-1991), **Ruler and the Round. Classic Problems in Geometrical Constructions**, Dover Publications, 2003, xi + 138 pages.
This is an unabridged republication of the 1970 book titled **Ruler and the Round or Angle Trisection and Circle Division**.

24 Alexander Aleksandrovich Kirillov (1967-), *Construction program. Regular polygons, Euler's function, and Fermat numbers*, Quantum **6** # 4 (March/April 1996), 10-15.
Article summary, posted at the URL just below: *On March 31, 1795, the great mathematician Carl Friedrich Gauss constructed the regular 17-gon and was so impressed that he embarked on a mathematical career. Later Gauss proved the constructibility of regular n-gons for all numbers n that are "Fermat primes." The author heads in the other direction and explains why regular*

polygons are constructible only for such values of n.
http://www.nsta.org/quantum/marcont.asp

25 Alexander Aleksandrovich Kirillov (1967-), *On regular polygons, Euler's function, and Fermat numbers*, pp. 87-98 in Serge [Sergei] L. Tabachnikov (editor), **Kvant Selecta: Algebra and Analysis I**, Mathematical World #14, American Mathematical Society, 1999.
This is essentially the same as the 1996 version, but the 1999 version contains a few more comments (especially in footnotes).

26 William Vernon Lovitt (1881-1972), **Elementary Theory of Equations**, Prentice-Hall, 1939, xi + 237 pages.
See Chapter XIV: *Ruler and Compass Constructions* (pp. 200-218). The Gauss method for solving $x^{17} = 1$ is sketched in Section 14.17: *Regular polygon of 17 sides* (pp. 214-216).

27 John Edward Maxfield (1927-) and Margaret Waugh Maxfield (1926-), **Abstract Algebra and Solution by Radicals**, Dover Publications, 1992, xi + 209 pages.
Highly recommended as an extremely gentle and well motivated (both in the examples given and in the exposition) introduction to modern algebraic notions that relate to constructible numbers and solvability by radicals. The Dover edition is a corrected reprint of the book first published in 1971.

28 Herbert Meschkowski (1909-1990), **Unsolved and Unsolvable Problems in Geometry**, translated by Jane Alexander Craig Burlak, Frederick Ungar Publishing Company (New York) and Oliver and Boyd (London), 1966, viii + 168 pages.
Chapter IX: *Constructions with Ruler and Compasses* (pp. 113-127) gives a nicely written, detailed, and relatively elementary discussion of its topic, including a proof (pp. 119-121) that a cubic equation with rational coefficients that has no rational solutions cannot have a constructible solution (application: $x^3 - 2 = 0$ is such an equation, so $\sqrt[3]{2}$ is not a constructible number).

29 Mark Daniel Meyerson (1949-), *The impossibility of trisecting angles*, Pi Mu Epsilon Journal **6** #10 (Spring 1979), 568-575.
What follows is from the 2nd paragraph. *Our goal is to give a brief and elementary proof of this nonconstructability. A few related theorems, such as the impossibility of duplicating the cube, are also included*. This paper is a very slight revision of a paper with the same title, published as: Modules and Monographs in Undergraduate Mathematics and its Applications, UMAP Unit 267, 30 April 1980.
http://www.eric.ed.gov/PDFS/ED218133.pdf

30 Aigli Helen Papantonopoulou (1947-), **Algebra: Pure and Applied**, Prentice-Hall, 2002, xx + 550 pages.
Chapter 11: *Geometric Constructions* (pp. 341-370) and Section 12.4: *Geometric Constructions Revisited* (pp. 415-422) give a nice general treatment of constructible numbers and many specific results involving cubic and quartic equations, including solvability of $x^n = 1$ by using quadratics *and* cubics on pp. 421-422 (related to pp. 421-422, see the material on *square root/cube root/trisection towers* on pp. 366-369). Chapter 13: *Historical Notes* (pp. 433-449) gives a survey

of the history of algebra, with a few remarks about Gauss and constructability on pp. 444-445.

31 James P. Pierpont (1866-1938), *On an undemonstrated theorem of the Disquisitiones Arithmeticæ*, Bulletin of the American Mathematical Society **2** #3 (December 1895), 77-83.
http://www.ams.org/journals/bull/1895-02-03/

32 Royal Society of London, **Catalogue of Scientific Papers, 1800-1900. Subject Index. Volume I: Pure Mathematics**, Cambridge University Press, 1908, lviii + 666 pages.
On pp. 222-224 (#2880) and p. 660 (#2880) there is a very complete list of pre-1900 references concerning solutions to $x^n = 1$. The titles of the references are often shortened and modified to only indicate content (in English), and full names of the cited journals are given at the beginning (pp. ix-xlix).
http://books.google.com/books?id=4BwPAAAAIAAJ&pg=222

33 Ian Nicholas Stewart (1945-), **Galois Theory**, 3rd edition, Chapman and Hall/CRC, 2004, xxxv + 288 pages.
This contains much of interest. See especially Chapter 21: *Circle Division* (pp. 233-250).

8. References for Solving $x^{17} = 1$

34 A. B. (??-??), *On the division of the circle into seventeen equal parts*, Philosophical Magazine (1) **57** #275 (March 1821), 172-176.
The introduction to this paper follows: *A curious discovery lately made in pure mathematics, we owe to M. Gauss of Göttingen, who has shown, contrary to the opinion that has prevailed from the most ancient times, that a regular polygon of seventeen sides may be inscribed in a circle, without having recourse to any other principles than those admitted in the plane geometry. As the author's own solution of this problem is a part of a peculiar and very abstruse and recondite theory, the following communication, in which the problem is solved without any reference to that theory, may not be unacceptable*.
http://books.google.com/books?id=BykwAAAAYAAJ&pg=PA172

35 Archibald [Archie] Brown (??-), *Trigonometric ratios in surd form*, Mathematical Gazette **68** #443 (March 1984), 48-51.
Using trigonometric and algebraic methods, Brown obtains explicit square root expressions for the values of $\cos\left(\frac{k\pi}{17}\right)$ for $k = 1,2,3$. To give an indication of Brown's approach, let $f(x) = x^2 - 2$. Then the image of $2\cos\left(\frac{2\pi}{17}\right)$ under f is $2\cos\left(\frac{4\pi}{17}\right)$, the image of $2\cos\left(\frac{4\pi}{17}\right)$ under f is $2\cos\left(\frac{8\pi}{17}\right)$, the image of $2\cos\left(\frac{8\pi}{17}\right)$ under f is $2\cos\left(\frac{16\pi}{17}\right)$, and the image of $2\cos\left(\frac{16\pi}{17}\right)$ under f is $2\cos\left(\frac{32\pi}{17}\right)$. Thus, $2\cos\left(\frac{32\pi}{17}\right)$ is a zero of the 16th degree polynomial $(f \circ f \circ f \circ f)(x)$. Now observe that $2\cos\left(\frac{32\pi}{17}\right)$ is equal to
$2\cos\left(\frac{32\pi}{17} - 2\pi\right) = 2\cos\left(-\frac{2\pi}{17}\right) = 2\cos\left(\frac{2\pi}{17}\right)$.

36 John Casey (1820-1891), **A Treatise on Plane Trigonometry, containing an Account of Hyperbolic Functions, with Numerous Examples**, Longmans, Green, and Company, 1888, xvi + 276 pages.
For a solution to $x^{17} = 1$ using trigonometric methods, see Exercises XXXVI on pp. 220-221. Also, some *Eratta* (none for pp. 220-221) is given on p. xvi.
http://books.google.com/books?id=78k2AAAAMAAJ&pg=PA220
http://www.archive.org/details/treatiseonplanet00caseuoft

37 Arthur Cayley (1821-1895), *On the equation* $x^{17} - 1 = 0$, Messenger of Mathematics **19** (1889-90), 184-188.
Reprinted on pp. 60-63 of Volume 13 (1897) of Cayley's **Collected Papers**.
http://books.google.com/books?id=YL8KAAAAIAAJ&pg=PA184
http://books.google.com/books?id=UXcNAQAAMAAJ&pg=PA60
http://name.umdl.umich.edu/ABS3153.0013.001

38 Claude Herries Chepmell (1864-1930), *Solution to Problem 2745*, American Mathematical Monthly **27** #7-9 (July-Sept. 1920), 331-332.
Explicit square root expressions for various lengths are given throughout a geometrical construction of $2\cos\left(\frac{2\pi}{17}\right)$.

39 Claude Herries Chepmell (1864-1930), *A solution of the binomial equation* $x^{17} = 1$, Mathematical Gazette **11** #156 (January 1922), 20-21.
This paper served as an outline for the solution I gave in great detail in Section 3.

40 Johann Carl Friedrich Gauss (1777-1855), **Disquisitiones Arithmeticae**, translated by Arthur A. Clarke and edited by William Charles Waterhouse, Yale University Press, 1966, xx + 472 pages.
An explicit square root expression for the value of $\cos\left(\frac{2\pi}{17}\right)$ is given on p. 454. The expression is essentially identical notationally in appearance to the expression given on p. 662 of the original 1801 Latin edition. This English translation was reprinted by Springer-Verlag in 1986. English translations of Articles 365-366 had previously appeared on pp. 348-350 of David Eugene Smith (1860-1944), **A Source Book in Mathematics**, McGraw-Hill Publishing Company, 1929, xvii + 701 pages [Reprinted by Dover Publications in 1959].

41 Émile Gelin (??-??), **Éléments de Trigonométrie Plane et Sphérique** [Elements of Plane and Spherical Trigonometry], Librairie Wesmael-Charlier (Namur, Belgium), 1888, ii + 252 pages.
See Articles 225-228 (pp. 230-235). The 2nd edition (288 pages) was published in 1906. Also of possible interest, on pp. 59-62 Gelin gives rationalized-denominator square root expressions for all six trigonometric functions for each of the angles 3°, 6°, 9°, ..., 87°. I believe square root expressions for the sine and cosine of all these angles may have been first obtained by Johann Heinrich Lambert. A table of the sine square root expressions, originally appearing in a 1770 publication of Lambert's, can be found on pp. 190-191 of Volume 1 of Lambert's **Opera Mathematica** (published in 1946). For the original 1770 publication where this table of values appears, see <http://books.google.com/books?id=fKhEAAAAcAAJ&pg=PA134>. It is known that if an integer n is not divisible by 3, then none of the six trigonometric functions evaluated at n° are constructible (or even expressible using only "real radicals"). For quite a few references about the notion of being expressible using only real radicals, see <http://tinyurl.com/4dz9vhx>.
http://books.google.com/books?id=1Qg5iGbPfLsC&pg=PA230

42 Robin Cope Hartshorne (1938-), **Geometry: Euclid and Beyond**, Undergraduate Texts in Mathematics, Springer-Verlag, 2000, xii + 526 pages.
See Chapter 6: *Construction Problems and Field Extensions* (pp. 241-294), especially Section 29: *The Regular 17-Sided Polygon* (pp. 250-259). Corollary 29.3 (p. 256) gives an explicit square root expression for the value of $2\sin\left(\frac{\pi}{17}\right)$, which is the side length of a regular 17-gon inscribed in the unit circle. (The coefficients of the three inner radicals were off by a factor of 2 in the first few printings of Hartshorne's book, but the expression was corrected by the 4th printing.)

43 Ernest William Hobson (1856-1933), **A Treatise on Plane and Advanced Trigonometry**, 7th edition, Dover Publications, 1957, xv + 383 pages.
Example 4 in Section 85 (p. 113 of the Dover edition; p. 111 of the 1st edition) gives an outline of

a trigonometric derivation for explicit square root expressions for the values of $\cos\left(\frac{2\pi}{17}\right)$ and $\sin\left(\frac{\pi}{17}\right)$. Incidentally, the Dover version is a reprint of the 7th edition (titled **A Treatise on Plane Trigonometry**). The 1st through 7th editions were originally published in 1891, 1897, 1911, 1918, 1921, 1925, and 1928.
http://books.google.com/books?id=_ktLAAAAMAAJ&pg=111
http://www.archive.org/details/treatiseonplanet00hobs
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44 Arthur Jones (1934-), Sidney Allen Morris (1947-); and Kenneth Robert Pearson (1943-), **Abstract Algebra and Famous Impossibilities**, Springer-Verlag, 1991, x + 187 pages.

45 Christian Felix Klein (1849-1925), **Famous Problems of Elementary Geometry**, translated by Wooster Woodruff Beman and David Eugene Smith, Dover Publications, 2003, xii + 92 pages.
Part I, Chapter IV (especially pp. 24-32) gives an algebraic derivation of an explicit square root expression for the value of $\cos\left(\frac{2\pi}{17}\right)$. The Gauss method is used and the treatment is fairly detailed. For many historical remarks that are related to statements that Klein made (or that Klein could have made), see Archibald [1] [2]. The Dover version is an unaltered reprint of the 1930 2nd English translation. The 1st English translation was published in 1897.
http://books.google.com/books?id=hOBEAAAAIAAJ
http://books.google.com/books?id=JNQ0q87OYBMC
http://name.umdl.umich.edu/ABN2381.0001.001

46 Victor Amédée Lebesgue (1791-1875), *Sur l'inscription des polygones réguliers de 15 et de 17 côtés* [On the inscription of regular polygons of 15 and 17 sides], Nouvelles Annales de Mathématiques (1) **5** (1846), 683-689.
This paper begins by applying the identity $\cos mx - \cos nx = -2\sin\left(\frac{m+n}{2}\right)\sin\left(\frac{m-n}{2}\right)$ to the equation $\cos 16x - \cos x = 0$, whose 15 nonzero solutions in the interval $0 \leq x < 2\pi$ are then found to be $\cos\left(\frac{2h\pi}{15}\right)$ and $\cos\left(\frac{2k\pi}{17}\right)$ for $h = 1, 2, \ldots, 7$ and $k = 1, 2, \ldots, 8$. Explicit square root expressions for several of these values are given.
http://books.google.com/books?id=-JpEAAAAcAAJ&pg=PA683
http://www.numdam.org/item?id=NAM_1846_1_5__683_1

47 A. M. M. (??-??), [untitled contribution to *Our Mathematical Column*], Knowledge: An Illustrated Magazine of Science Plainly Worded-Exactly Described **3** #82 (25 May 1883), 316-317.
This gives a purely algebraic solution to $x^{17} = 1$.
http://www.archive.org/details/knowledgev140nov03londuoft

48 Dennis Gerard Creaser McKeon (??-) and Thomas Niall Sherry (??-), *Exploring cyclotomic polynomials*, Mathematical Gazette **85** #502 (March 2001), 59-65.
This paper has several aspects that merit inclusion in our bibliography. One example is a geometric figure (p. 64) based on sequence of 4 non-overlapping triangles in the interior of a right triangle from which the value of $\cos\left(\frac{\pi}{17}\right)$ can be obtained. I believe similar constructions can be found on pp. 226-228 of August Adler's book **Theorie der Geometrischen Konstruktionen** (1906) and on pp. 65-66 of Robert Goldenring's pamphlet **Die Elementargeometrischen Konstruktionen des Regelmässigen Siebzehnecks** (1915). Adler's book is freely available on the internet at <http://books.google.com/books?id=PeFUAAAAYAAJ> and Goldenring's pamphlet is freely available on the internet at <http://books.google.com/books?id=xFzvAAAAMAAJ>.

49 Joseph Alfred Serret (1819-1885), **Traité de Trigonométrie** [Treatise on Trigonometry], 6th edition, Gauthier-Villars, 1880, x + 336 pages.
For a trigonometric solution to $x^{17} = 1$, see Article 174 (pp. 216-219). The material that specifically pertains to $x^{17} = 1$ does not appear in the 1850 1st edition. However, this material, and the pages it appears on, seems to be unchanged in the 7th and 8th editions. The 1st through 9th editions were originally published in 1850, 1857, 1862, 1868, 1875, 1880, 1888, 1900, and 1908.
http://name.umdl.umich.edu/ABN8403.0001.001
http://books.google.com/books?id=ZapPAAAAYAAJ&pg=PA216
http://books.google.com/books?id=3IQAAAAAMAAJ&pg=PA216

50 John Edward Aloysius Steggall (1855-1935), *The value of* $\cos 2\pi/17$ *expressed in quadratic radicals*, Proceedings of the Edinburgh Mathematical Society **7** (1889), 4-5.
Steggall states the following in a footnote at the beginning: *This paper is merely intended to show how the solution of this interesting case of the binomial equation may be exhibited in a form suitable for a course of Elementary Trigonometry*.
http://books.google.com/books?id=sZrxAAAAMAAJ&pg=RA2-PA4

51 Heinrich Franz Friedrich Tietze (1880-1964), **Famous Problems of Mathematics. Solved and Unsolved Mathematical Problems From Antiquity to Modern Times**, translated and edited by Beatrice Kevitt Hofstadter and Horace Komm, Graylock Press, 1965, xvi + 367 pages.
See Chapter IX: *The Regular Polygon of 17 Sides* (pp. 182-199) and *Notes To Chapter IX* (pp. 200-210).

52 Charles Edgar White (1868-1943), **Theory of Irreducible Cases of Equations and its Applications in Algebra, Geometry and Trigonometry**, Part 2, published by Charles Edgar White (Press of The New Era Printing Company), 1913, v + 90 pages.
See especially Articles 37-41 (pp. 66-75).
http://books.google.com/books?id=EcZLAAAAYAAJ&pg=PA66

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53 Wayne Wilson Bishop (1942-), *How to construct a regular polygon*, American Mathematical Monthly **85** #3 (March 1978), 186-188.
http://steiner.math.nthu.edu.tw/disk5/js/geometry/bishop.pdf
http://apollonius.math.nthu.edu.tw/d1/ne01/jyt/linkjstor/regular/5.pdf

54 Arthur Cayley (1821-1895), *Note sur la solution de l'équation* $x^{257} - 1 = 0$ [Note on the solution of the equation $x^{257} - 1 = 0$], Journal für die reine und angewandte Mathematik [= Crelle's Journal] **41** (1851), 81-83.
Reprinted on pp. 564-566 of Volume 1 (1889) of Cayley's **Collected Papers**.
http://books.google.com/books?id=hQoPAAAAIAAJ&pg=PA81
http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN243919689_0041
http://books.google.com/books?id=zJdQAAAAYAAJ&pg=PA564
http://name.umdl.umich.edu/ABS3153.0001.001

55 Duane William DeTemple (??-), *Carlyle circles and the Lemoine simplicity of polygon constructions*, American Mathematical Monthly **98** #2 (February 1991), 97-108.
http://apollonius.math.nthu.edu.tw/d1/ne01/jyt/linkjstor/regular/1.pdf

56 Peter Alexander Fischer (1807-1867), *Resolutio algebraica aequationis* $x^{257} - 1 = 0$ [Solution to the algebraic equation $x^{257} - 1 = 0$], Journal für die reine und angewandte Mathematik [= Crelle's Journal] **11** (1834), 201-218.
http://books.google.com/books?id=YQUPAAAAIAAJ&pg=PA201
http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN243919689_0011

57 Jacob Badon Ghijben [Ghyben] (1798-1870), *Beschouwing van den regelmatigen 257-hoek*, [Considerations on the regular 257-gon], Verslagen en Mededeelingen der Koninklijke Akademie van Wetenschappen, Afdeeling Natuurkunde (Amsterdam) (2) **2** (1868), 1-34
http://books.google.com/books?id=pyLNAAAAMAAJ&pg=PA1

58 Christian Gottlieb (??-), *The simple and straightforward construction of the regular 257-gon*, Mathematical Intelligencer **21** #1 (Winter 1999), 31-37.

59 Karl Hagge (??-1932), *Einfache behandlung der 257-teilung des kreises* [Simple treatment of the 257-division of the circle], Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht **41** (1910), 448-458.

60 Ernesto Pascal (1865-1940), *Sulla costruzione del poligono regolare di 257 lati* [On the construction of a regular polygon of 257 sides], Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche (Sezione della Società Reale di Napoli) (2) **1** (1887), 33-39.
http://books.google.com/books?id=TcYOAQAAMAAJ&pg=PA33

61 Friedrich Julius Richelot (1808-1875), *De resolutione algebraica aequationis* $x^{257} = 1$, *sive de divisione circuli per bisectionem anguli septies repetitam in partes 257 inter se aequales commentatio coronata* [A thesis on the algebraic solution of the equation $x^{257} = 1$, or on the division of the circle into 257 equal parts through a sevenfold iteration of the bisection of an angle], Journal für die reine und angewandte Mathematik [= Crelle's Journal] **9** (1832), 1-26 & 146-161 & 209-230 & 337-358.
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62 Julius [Gyula] Strommer (1920-1995), *Konstruktion des regulären 257-ecks mit lineal und streckenübertrager* [Construction of the regular 257-gon with straightedge and segment transfer], Acta Mathematica Hungarica **70** (1996), 259-292.
A short description of this paper (from Mathematical Reviews) can be found at <http://www.math.niu.edu/~rusin/known-math/97/construct>.

63 Michael Trott (??-), $\cos(2\pi/257)$ *à la Gauss*, Mathematica in Education and Research **4** #2 (1995), 31-36.
http://library.wolfram.com/infocenter/Articles/1693/