$$\left.\frac{df}{dx}\right]_{x=\theta(p)} = f(p). \tag{E}$$

Finally, if the function $\alpha(x, z)=\theta(x)+z$ satisfies the conditions laid down for a differentiator in §2, we can reduce (E) to

$$f_\alpha(x) = f(x), \tag{F}$$

using again the lemma demonstrated at the beginning of this paragraph. In many cases, solving (F) may be simpler than solving (D) directly; it is clear that any solution of (F) will also be a solution of (D).

ON INFINITE RADICALS

By AARON HERSCHFELD, Columbia University

Introduction. For approximately twenty-five years Professor Edward Kasner has periodically suggested to his classes at Columbia University the investigation of the problem of "infinite radicals." Thus it was proposed to find conditions for the convergence or divergence of the "right" infinite radical

$$\sqrt{a_1+\sqrt{a_2+\sqrt{a_3+\cdots}}}$$

and the "left" infinite radical

$$\cdots\sqrt{a_3+\sqrt{a_2+\sqrt{a_1}}}.$$

In particular, what are the properties of the number K (which we shall call the Kasner number)

$$K = \sqrt{1+\sqrt{2+\sqrt{3+\cdots}}}\ ? \tag{1}$$

Vieta is probably the first to have used finite radicals of an arbitrary number of root extractions. His famous infinite product, the first purely arithmetical process for calculating π, may be written*

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\cdots\cdot \tag{2}$$

Before passing on to twentieth century work we may mention the following brief quotation from *The Life-Romance of an Algebraist* by George Winslow Pierce, 1891, p. 18:

"When a boy I discovered

$$\pi = 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots}}} \qquad (\text{with } n(=\infty)\sqrt{\ }\text{'s}), \tag{3}$$

$$\pi^{2^n} - \cdots = 2^{2^n\cdot n+1},$$

* Glaisher, Messenger of Mathematics, vol. 2, (new series), 1873, p. 124.

and showed it to a friend, in the American Nautical Almanac office, Simon Newcomb,—an illustration of infinite IT."

1. *Ramanujan's Problem.* In April, 1911, Srinivasa Ramanujan published the following problem:*

Find the value of:

(i) $$\sqrt{}[1 + 2\sqrt{}\{1 + 3\sqrt{}(1 + \cdots)\}],$$

(ii) $$\sqrt{}[6 + 2\sqrt{}\{7 + 3\sqrt{}(8 + \cdots)\}].$$

Ramanujan's solution† is incomplete, as we shall now show.

His solution of (i) is as follows:

$$f(n) \equiv n(n+2) = n\sqrt{1 + (n+1)(n+3)} = n\sqrt{1 + f(n+1)},$$
$$f(n) = n\sqrt{1 + (n+1)\sqrt{1 + f(n+2)}} = \cdots \quad \cdots,$$
$$f(1) = 3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}.$$

This solution is incomplete, for we may write similarly

$$4 = \sqrt{1 + 2\cdot(15/2)} = \sqrt{1 + 2\sqrt{1 + 3\cdot(221/12)}} = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}},$$

and thus obtain the value 4 for the expression (i).

Thus Ramanujan has shown that

(4) $$\begin{aligned} 3 &= \sqrt{1 + 2\cdot 4} = \sqrt{1 + 2\sqrt{1 + 3\cdot 5}} = \cdots \\ &= \sqrt{1 + 2\sqrt{\cdots + n\sqrt{1 + (n+1)(n+3)}}} \end{aligned}$$

and has concluded, without giving adequate justification, that the sequence

(5) $$\sqrt{1},\ \sqrt{1 + 2\sqrt{1}},\ \cdots,\ \sqrt{1 + 2\sqrt{1 + \cdots + n\sqrt{1}}},\ \cdots$$

converges to the limit 3. But this can be proved as follows.

It is clear, upon comparing (4) and (5), that the sequence (5) is monotonic and bounded (<3). Hence it converges to a limit $\leqq 3$. To show that the limit is exactly 3 we shall prove that given an arbitrarily small positive $\epsilon < 3$, there exists an integer $N = N(\epsilon)$ such that for all $n > N$,

(6) $$u_n \equiv \sqrt{1 + 2\sqrt{1 + \cdots + n\sqrt{1}}} > 3 - \epsilon.$$

Since $3 - \epsilon = 3r$, where $0 < r = 1 - \epsilon/3 < 1$, we are to show that

$$u_n > 3r = r\sqrt{1 + 2\sqrt{1 + \cdots + n\sqrt{1 + (n+1)(n+3)}}},$$

that is,

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* Journal of the Indian Mathematical Society, vol. 3 (1911), p. 90, problem 289.

† *Ibid.*, vol. 4 (1912), p. 226. Reproduced in G. H. Hardy's *Collected Papers of Srinivasa Ramanujan*, page 323.

$$\sqrt{1+2\sqrt{1+\cdots+n\sqrt{1}}} > \sqrt{r^2+2\sqrt{r^{2^2}+\cdots+n\sqrt{r^{2^n}[1+(n+1)(n+3)]}}}.$$

But $1>r^{2^i}$, $i=1, 2, \cdots$, and there exists an integer N_1 such that for all $n>N_1$

$$1 > r^{2^n}[1+(n+1)(n+3)] = r^{2^n}(n+2)^2,$$

since $r^{2^n}(n+2)^2 \to 0$ as $n\to\infty$. Hence we may take $N=N_1$ in inequality (6) and our proof is completed. In like manner we can complete Ramanujan's proof that the expression (ii) is equal to 4.

2. *Pólya's Criterion.* We return now to the problem of the convergence or divergence of the "right" infinite radical, i.e., the behavior of the sequence $\{u_n\}$ where

$$u_n = \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}.$$

This problem was proposed by G. Pólya* in the following reduced form:

Show that the sequence $\{u_n\}$

$$\left.\begin{matrix}\text{converges}\\ \text{diverges}\end{matrix}\right\} \text{ if } \overline{\lim_{n\to\infty}} \frac{\log\log a_n}{n} \begin{matrix}<\\ >\end{matrix} \log 2.$$

It is to be understood that $a_n \geqq 0$, all square roots are taken positive, and that for $a_n \leqq 1$, we shall adopt the convention of Pólya and Szegö that $(\log\log a_n)/n \equiv -\infty$.

Inasmuch as this statement of the problem says nothing concerning the case $\overline{\lim}_{n\to\infty}(\log\log a_n)/n = \log 2$, it is of interest to note that it is then necessary and sufficient for convergence that

$$\overline{\lim_{n\to\infty}}\, n\left\{\frac{\log\log a_n}{n} - \log 2\right\} < +\infty, \tag{7}$$

that is, that there exist an upper limit (7), either finite or $=-\infty$. This may be deduced simply from a necessary and sufficient condition for the convergence of the sequence $\{u_n\}$, which is apparently new and which we now derive.

THEOREM I. The sequence $\{u_n\}$ defined by

$$u_n \equiv \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}},$$

converges if and only if there exists a *finite* upper limit

$$\overline{\lim_{n\to\infty}}\, a_n^{2^{-n}} < +\infty.$$

Proof. First, suppose $\{u_n\}$ converges. Since

$$u_n \geqq \sqrt{0+\sqrt{a_2+\cdots+\sqrt{a_n}}} \geqq \cdots \geqq \sqrt{0+\sqrt{0+\cdots+\sqrt{a_n}}} = a_n^{2^{-n}}$$

the $\overline{\lim}_{n\to\infty} a_n^{2^{-n}}$ must be finite, i.e. $\{a_n^{2^{-n}}\}$ is bounded.

* Aufgabe; Arch. d. Math. u. Phys. Serie 3, Bd. 24 (1916), p. 84: see also Pólya and Szegö *Aufgaben und Lehrsätze*, vol. 1, p. 30, problem 162.

Second, suppose $\overline{\lim}_{n\to\infty} a_n^{2^{-n}} < +\infty$. Then we may select a $G>0$ such that, for all $n>0$, $a_n^{2^{-n}} \leqq G$ and thus

$$a_n \leqq G^{2^n}.$$

Hence

$$\begin{aligned} u_n &\leqq \sqrt{G^2 + \sqrt{G^{2^2} + \cdots + \sqrt{G^{2^n}}}} \\ &= G\sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}. \end{aligned}$$

But

$$2 = \sqrt{2+2} = \sqrt{2+\sqrt{2+2}} = \cdots = \sqrt{2+\sqrt{2+\cdots+\sqrt{2+2}}} > \sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}$$

and thus

$$u_n < 2G, \text{ for all } n > 0.$$

However the sequence $\{u_n\}$ is non-decreasing; therefore it converges.

Let us now consider the criterion given by M. Pólya. In the first of the two cases $\overline{\lim}_{n\to\infty} (\log \log a_n)/n < \log 2$, that is, we may find an N such that for all $n>N$, $(\log \log a_n)/n < \log 2$, $a_n^{2^{-n}} < e$, and thus

$$\overline{\lim_{n\to\infty}}\, a_n^{2^{-n}} \leqq e.$$

But in the second case $\overline{\lim}_{n\to\infty} (\log \log a_n)/n > \log 2$. Hence for some number $a>1$, $(\log \log a_n)/n > a \log 2$, i.e., $\log a_n > 2^{an}$, for an infinite number of values of n. That is, for these values of n,

$$a_n^{2^{-n}} > e^{2^{(a-1)n}},$$

and thus

$$\overline{\lim_{n\to\infty}}\, a_n^{2^{-n}} = +\infty.$$

Finally there is the case $\overline{\lim}_{n\to\infty} (\log \log a_n)/n = \log 2$. Suppose $\{u_n\}$ converges in this case. Then $\overline{\lim}_{n\to\infty} a_n^{2^{-n}}$ is finite, by theorem I, i.e., there exists a $G>1$ such that $a_n^{2^{-n}} < G$ for all n. Hence $\log a_n < 2^n \log G$ and if $a_n > 1$

$$n\{(\log \log a_n)/n - \log 2\} < \log \log G$$

while if $a_n \leqq 1$ we have, by convention, $(\log \log a_n)/n = -\infty$. Consequently the condition (7) that $\overline{\lim}_{n\to\infty} n\{\log \log a_n)/n - \log 2\} < +\infty$ is necessary for convergence. It is also sufficient. For suppose the condition (7) holds but $\{u_n\}$ does not converge. Then $\overline{\lim}_{n\to\infty} a_n^{2^{-n}} = +\infty$ and we may find, given any $G>e$, an infinitude of integers n such that $a_n^{2^{-n}} > G$ and thus $\log a_n > 2^n \log G$ so that

$$n\{(\log \log a_n)/n - \log 2\} > \log \log G.$$

This contradicts our hypothesis that the condition (7) holds.

3. *Degree of Approximation.* It is easy to apply our method for completing the solution of Ramanujan's problem to the proof that $2=\sqrt{2+\sqrt{2+\cdots}}$. But we shall derive a more interesting relation which gives us an insight into the degree of approximation of $u_n=\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}$. Thus

$$2\cos(\theta/2) = \sqrt{2+2\cos\theta} = \sqrt{2+\sqrt{2+2\cos 2\theta}} = \cdots$$
$$= \sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos 2^{n-1}\theta}}}$$

provided $\cos(\theta/2)\geqq 0$, $\cos\theta\geqq 0, \cdots, \cos 2^{n-2}\theta\geqq 0$, (we consider only non-negative square roots). But this condition is satisfied for $\theta=\pi/2^n$ and so

$$2\cos\frac{\pi}{2^{n+1}} = \sqrt{2+\sqrt{2+\cdots\sqrt{2+0}}} = u_n.$$

Hence $\lim_{n\to\infty} u_n=2$ and $2-u_n=2[1-\cos(\pi/2^{n+1})]$. Therefore

$$\lim_{n\to\infty} 2^{2n}(2-u_n) = \pi^2/4, \qquad 2-u_n \sim \pi^2/(4\cdot 2^{2n}).$$

Since

$$\sqrt{2-u_{n-1}} = \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \sim \pi/2^n,$$

where u_n has n radicals, it is clear that equation (3) of G. W. Pierce may be given a rigorous interpretation. But it is not as clear just what the second of Pierce's equations really means.

Consider now the more general case (where every $a_n>0$) and write

$$u_n = \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}$$
$$U_n = \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n+r_n}}} \qquad r_n \geqq 0.$$

We seek an approximate formula for the difference U_n-u_n. We shall make repeated application of the formula, easily proved,

$$\sqrt{a+x} \leqq \sqrt{a}+\frac{x}{2\sqrt{a}}, \qquad a>0\leqq x.$$

Thus

$$\sqrt{a_n+r_n} \leqq \sqrt{a_n}+\frac{r_n}{2\sqrt{a_n}}$$

$$\sqrt{a_{n-1}+\sqrt{a_n+r_n}} \leqq \sqrt{a_{n-1}+\sqrt{a_n}+\frac{r_n}{2\sqrt{a_n}}} \leqq \sqrt{a_{n-1}+\sqrt{a_n}}$$
$$+\frac{r_n}{2^2\sqrt{a_n}\cdot\sqrt{a_{n-1}+\sqrt{a_n}}}$$

. . . .

$$\sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n+r_n}}} \leqq \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}$$
$$+\frac{r_n}{2^n\sqrt{a_n}\cdot\sqrt{a_{n-1}+\sqrt{a_n}}\cdot\ \cdots\ \cdot\sqrt{a_1+\cdots+\sqrt{a_n}}}$$

(8)
$$U_n - u_n \leqq \frac{r_n}{2^n\sqrt{a_n}\cdot\sqrt{a_{n-1}+\sqrt{a_n}}\cdot\ \cdots\ \cdot\sqrt{a_1+\cdots+\sqrt{a_n}}}.$$

This inequality is obviously sharper than that given by Pólya and Szegö,* namely

$$U_n - u_n < \frac{r_n}{2^n\sqrt{a_n}\cdot\sqrt{a_{n-1}}\cdot\ \cdots\ \cdot\sqrt{a_1}}.$$

In the particular case $u_n=\sqrt{2+\sqrt{\ \cdots\ +\sqrt{2}}}$, $r_n=2$, we have $U_n=2$. From the inequality in the *Aufgaben* we may infer merely

$$2-u_n = U_n - u_n < \frac{2}{2^n\cdot\sqrt{2^n}} = \frac{2}{2^{3n/2}}.$$

From inequality (8) however we get

$$2-u_n = U_n - u_n \leqq \frac{2}{2^n\sqrt{2}\cdot\sqrt{2+\sqrt{2}}\cdot\ \cdots\ \cdot\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}.$$

Now $u_n=\sqrt{2+\ \cdots\ +\sqrt{2}}=2\cos(\pi/2^{n+1})$ so that $\sqrt{2}\cdot\sqrt{2+\sqrt{2}}\cdot\ \cdots\ \cdot\sqrt{2+\ \cdots\ +\sqrt{2}}=2^n\cos(\pi/2^2)\cdot\cos(\pi/2^3)\cdot\ \cdots\ \cdot\cos(\pi/2^{n+1})$. Hence

$$2-u_n = U_n - u_n \leqq \frac{2}{2^n u_1 u_2\cdots u_n} \sim \frac{2}{2^{2n}\cdot 2/\pi} = \frac{\pi}{2^{2n}}$$

which gives the correct order of smallness of $2-u_n \sim \pi^2/4\cdot 2^{2n}$.

4. *Examples of Infinite Radicals.* Let us examine now the infinite radical defined by $u_n=\sqrt{x+\sqrt{x+\ \cdots\ +\sqrt{x}}}$, $x>0$. The sequence $\{u_n\}$ is monotone increasing. For $x\leqq 1$ it is bounded by the corresponding sequence for $x=1$ which, as we saw earlier, is bounded by 2. And when $x>1$,

$$u_n < \sqrt{x^2+\sqrt{x^{2^2}+\cdots+\sqrt{x^{2^n}}}} = x\sqrt{1+\sqrt{1+\cdots+\sqrt{1}}} < 2x.$$

Hence we have established, independently of theorem I, the convergence of $\{u_n\}$. But, calling $\lim_{n\to\infty} u_n=u$, we may calculate u thus: $u_{n+1}^2=x+u_n$ and therefore

* *Aufgaben und Lehrsätze,* vol. 1, p. 30, problem 163.

$$u^2 = x + u, \qquad u = \frac{1+\sqrt{1+4x}}{2}, \qquad x > 0.$$

However

$$u = 0 \quad \text{if } x = 0.$$

We are now able to compute Kasner's number K which exists by Theorem I. Thus

$$K > \sqrt{1+\sqrt{2+\cdots+\sqrt{9+\sqrt{10+\sqrt{10+\cdots}}}}} = 1\cdot 757933\cdots$$

$$K < \sqrt{1+\sqrt{\cdots+\sqrt{9+\sqrt{10+\sqrt{10^2+\sqrt{10^{2^2}+\cdots}}}}}} = 1\cdot 757933\cdots$$

$$K = 1\cdot 757933\cdots.$$

In our calculation we used Bruns' seven place logarithms (1894).

Consider now the relation

$$x(2^n + x) = x\sqrt{2^{2n} + x(2^{n+1}+x)}.$$

Thus

$$x(2+x) = x\sqrt{2^2 + x(2^2+x)} = x\sqrt{2^2 + x\sqrt{2^4 + x(2^3+x)}} = \cdots$$
$$= x\sqrt{2^2 + x\sqrt{2^4 + \cdots + x\sqrt{2^{2n} + x(2^{n+1}+x)}}}.$$

It is easy to prove (as in Ramanujan's problem) that

(9) $$x(2+x) = x\sqrt{2^2 + x\sqrt{2^4 + x\sqrt{\cdots}}}$$

and so

$$3 = \sqrt{2^2 + \sqrt{2^4 + \sqrt{\cdots + \sqrt{2^{2n} + \cdots}}}}.$$

From (9) we have, substituting $x/2$ for x,

$$2\left(1+\frac{x}{4}\right) = \sqrt{2^2 + \frac{x}{2}\sqrt{2^4 + \frac{x}{2}\sqrt{2^6+\cdots}}}$$

and

$$\left(1+\frac{x}{4}\right) = \sqrt{1+\frac{x}{2}\sqrt{1+\frac{x}{2^2}\sqrt{1+\cdots+\sqrt{1+\frac{x}{2^n}\sqrt{\cdots}}}}}.$$

Left infinite radicals satisfy a characteristic difference equation of the second degree. Thus

$$u_n = \sqrt{a_n + \sqrt{\cdots + \sqrt{a_1}}}, \qquad u_{n+1} = \sqrt{a_{n+1} + \sqrt{a_n + \cdots + \sqrt{a_1}}}$$

$$u_{n+1}^2 = a_{n+1} + u_n.$$

On the other hand, in the case of right infinite radicals, if $\{u_n\}$ converges to a limit u,

$$u_n = \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}}$$

$$u = \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n + r_n}}}$$

$$r_n = \sqrt{a_{n+1} + r_{n+1}}, \qquad r_{n+1} = r_n^2 - a_{n+1}.$$

5. *Left Infinite Radicals.* We now prove

THEOREM II. The necessary and sufficient condition for the convergence of $\{u_n\}$,

$$u_n = \sqrt{a_n + \sqrt{a_{n-1} + \cdots + \sqrt{a_1}}},$$

is that there exist a limit a of the sequence $\{a_n\}$, i.e., that $a_n \to a$. When such a limit exists then

$$u_n \to \frac{1 + \sqrt{1 + 4a}}{2} \qquad \text{if } a > 0$$

$$u_n \to 1 \qquad \text{if } a = 0 \text{ and at least one } a_n > 0$$

$$u_n \equiv 0 \qquad \text{if } a = 0 \text{ and all } a_n = 0.$$

Proof. First suppose the radical converges. Observe that

$$u_{n+1}^2 = a_{n+1} + u_n \quad \text{and} \quad a_{n+1} = u_{n+1}^2 - u_n \to u^2 - u$$

where u_n converges to the limit u. Hence the condition is necessary.

Next suppose that $a_n \to a > 0$, assuming that each $a_n \geqq 0$. Given any positive $\delta < a$ there exists an integer N such that for all $n > N$,

$$0 < a - \delta < a_n < a + \delta.$$

Hold δ fixed and choose a positive ϵ arbitrarily small. There exists an integer N_1 such that

$$(1 + \epsilon)^{2^{N_1}} > \frac{a + \delta + u_N}{a + \delta}, \qquad N = N(\delta) \text{ and } N_1 = N_1(\delta,\epsilon).$$

Hence for all $n > N + N_1$

$$u_n = \sqrt{a_n + \cdots + \sqrt{a_{N+1} + u_N}} < \sqrt{a + \delta + \sqrt{\cdots + \sqrt{a + \delta + u_N}}}$$
$$< (1 + \epsilon)\sqrt{a + \delta + \sqrt{\cdots + \sqrt{a + \delta}}}$$
$$< (1 + \epsilon)\sqrt{a + \delta + \sqrt{a + \delta + \sqrt{\cdots}}}.$$

Again,

$$u_n > \sqrt{a - \delta + \sqrt{\cdots + \sqrt{a - \delta + u_N}}} \geqq \sqrt{a - \delta + \sqrt{\cdots + \sqrt{a - \delta}}}$$

where the last expression involves $n-N$ radicals. Letting $n\to\infty$,

$$\sqrt{a-\delta+\sqrt{a-\delta+\sqrt{\cdots}}} \leqq \varliminf_{n\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n$$

$$\leqq (1+\epsilon)\sqrt{a+\delta+\sqrt{a+\delta+\sqrt{\cdots}}}$$

i.e.,

$$\frac{1+\sqrt{1+4(a-\delta)}}{2} \leqq \varliminf_{n\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n \leqq (1+\epsilon)\cdot\frac{1+\sqrt{1+4(a+\delta)}}{2}.$$

Now letting $\epsilon\to 0$,

$$\frac{1+\sqrt{1+4(a-\delta)}}{2} \leqq \varliminf_{n\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n \leqq \cdot\frac{1+\sqrt{1+4(a+\delta)}}{2}.$$

But $\delta>0$ is arbitrary and may be permitted to approach zero. Thus

$$\frac{1+\sqrt{1+4a}}{2} \leqq \varliminf_{m\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n \leqq \frac{1+\sqrt{1+4a}}{2}$$

$$\lim_{n\to\infty} u_n = \frac{1+\sqrt{1+4a}}{2}.$$

Let us now consider the case $a_n\to a=0$. If every $a_n=0$ then every $u_n=0$ and so $u=0$. We may therefore suppose at least one $a_n>0$ and, without any loss of generality, we let $a_1>0$. Thus for any $n\geqq 1$

$$u_n = \sqrt{a_n+\sqrt{\cdots+\sqrt{a_1}}} \geqq a_1^{2^-}$$

and thus

$$\varliminf_{n\to\infty} u_n \geqq 1.$$

As before, given any arbitrarily small positive δ we may find an integer $N_1=N_1(\delta)$ such that for all $n>N_1$, $a_n<\delta$. Hold δ fixed and select an $\epsilon>0$ arbitrarily small. To ϵ there corresponds an integer N such that for $n>N$,

$$(1+\epsilon)^{2^n} > (1+\epsilon)^{2^N} > \frac{\delta+u_{N_1}}{\delta}, \qquad N = N(\delta,\epsilon).$$

Hence for all $n>N+N_1$

$$u_n < \sqrt{\delta+\sqrt{\cdots+\sqrt{\delta+u_{N_1}}}} < (1+\epsilon)\sqrt{\delta+\sqrt{\cdots+\sqrt{\delta}}}$$

$$< (1+\epsilon)\sqrt{\delta+\sqrt{\delta+\sqrt{\cdots}}}$$

$$= (1+\epsilon)\cdot\frac{1+\sqrt{1+4\delta}}{2}.$$

Letting $\epsilon \to 0$

$$\overline{\lim_{n\to\infty}}\, u_n \leqq \frac{1+\sqrt{1+4\delta}}{2}.$$

Since $\delta > 0$ is arbitrarily small we may let $\delta \to 0$ and so

$$1 \leqq \underline{\lim_{n\to\infty}}\, u_n \leqq \overline{\lim_{n\to\infty}}\, u_n \leqq 1, \qquad \lim_{n\to\infty} u_n = 1.$$

6. *Generalized Infinite Radicals.*

It is interesting to notice that if we generalize right infinite radicals by writing

$$u_n = \left\{a_1 + \left[a_2 + (a_3 + \cdots + a_n^{r_n})^{r_3}\right]^{r_2}\right\}^{r_1}$$

we may secure the ordinary series form by putting $r_1 = r_2 = \cdots = 1$, i.e.,

$$u_n = a_1 + a_2 + \cdots + a_n.$$

Also if we put all the exponents $= -1$ we obtain the continued fraction

$$u_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}}.$$

Through a slightly broader generalization of right infinite radicals we may obtain as special cases the infinite product and the ascending continued fraction.

One may readily prove

THEOREM III. Let

$$u_n = (a_1 + \{a_2 + \cdots + a_n^{r_n}\}^{r_2})^{r_1} \tag{1}$$

where $a_i \geqq 0$, $0 < r_i \leqq 1$, $i = 1, 2, \cdots$, and the series

$$S = \sum_{i=1}^{\infty} r_1 r_2 \cdots r_i \tag{2}$$

converges. Then the necessary and sufficient condition for the sequence $\{u_n\}$ defined by (1) to converge is that

$$\overline{\lim_{n\to\infty}}\, a_n^{r_1 r_2 \cdots r_n} < +\infty.$$

It is also of interest to generalize infinite radicals to include negative or complex elements a_n. Convergence questions appear to become very difficult in such cases.

In addition to the references mentioned in this article the following may be noted:

1. "*Aufgaben und Lehrsätze*,"—G. Pólya and G. Szegö, vol. 1, p. 29, problem 161; vol. 1, p. 33, problems 183–5; the solutions refer to other literature.

2. Jahrbuch Fortschr. d. Math., vol. 39 (1908), p. 501; M. Cipolla, "*Intorno ad un radicale continuo*"; G. Candido, "*Sul numero π*."

3. Jahresbericht d. Deutsch Math. Ver., vol. 33 (1924), p. 69 and pp. 117–118; vol. 39 (1930), p. 6.

4. Zeitschr. f. Math. und naturw. Unterr., vol. 41 (1910), p. 161–186.

5. American Math. Monthly, Jan. 1917, problem 460.

6. National Mathematics Magazine, (Baton Rouge, La.), April 1935, problems 75, 78; May 1935, p. 247–8, 251–2.

ON SPHERES ASSOCIATED WITH THE TETRAHEDRON

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Notations. Let $ABCD$ be a tetrahedron with edges $BC=a$, $CA=b$, $AB=c$, $DA=a'$, $DB=b'$, $DC=c'$; (O) its circumsphere with center O and radius R; G its centroid; G_a, G_b, G_c, G_d the centroids of the faces BCD, CDA, DAB, ABC respectively; Ω the Monge point† (i.e., the symmetric of O with respect to G); M_a, M_b, M_c, M_d the medians of the tetrahedron (i.e., line segments which join the vertices A, B, C, D to the centroids G_a, G_b, G_c, G_d of the opposite faces).

1. We point out some little known formulas which are useful in the study of the tetrahedron.

We have first‡

$$(1)\quad \begin{aligned} M_a^2 &= \overline{AG_a}^2 = (a'^2 + b^2 + c^2)/3 - (a^2 + b'^2 + c'^2)/9 \\ M_b^2 &= \overline{BG_b}^2 = (a^2 + b'^2 + c^2)/3 - (a'^2 + b^2 + c'^2)/9 \\ M_c^2 &= \overline{CG_c}^2 = (a^2 + b^2 + c'^2)/3 - (a'^2 + b'^2 + c^2)/9 \\ M_d^2 &= \overline{DG_d}^2 = (a'^2 + b'^2 + c'^2)/3 - (a^2 + b^2 + c^2)/9; \end{aligned}$$

from which we obtain§

$$(2)\quad M_a^2 + M_b^2 + M_c^2 + M_d^2 = 4(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/9,$$

and then

$$(3)\quad \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2 = (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/4.$$

We shall call the line segments MN, ST, UV joining the midpoints of the

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* Translated by J. E. LaFon, University of Oklahoma.

† Monge, Correspondance de l'Ecole Polytechnique de Paris, t. II.

‡ See Educational Times, 1890, p. 114. Journal de G. de Longchamps, 1890, p. 262. V. Thébault, Mathesis, 1930, p. 254.

§ V. Thébault, loc. cit.